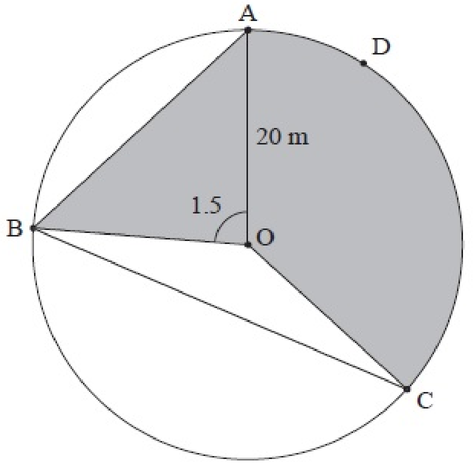
Topic 3 Review *[82 marks]*

The following diagram shows a circular play area for children.



The circle has centre O and a radius of 20 m, and the points A, B, C and D lie on the circle. Angle AOB is 1.5 radians.

1a. Angle BOC is 2.4 radians. *[3 marks]*

Find the length of arc ADC.

Markscheme

appropriate method to find angle AOC *(M1)*

e.g. $2\pi - 1.5 - 2.4$

correct substitution into arc length formula *(A1)*

e.g. $(2\pi - 3.9) \times 20$, $2.3831853\ldots \times 20$

arc length $= 47.6637\ldots$

arc length $= 47.7$ $(47.6, 47.7]$ (i.e. do **not** accept 47.6) ***A1*** ***N2***

Notes: Candidates may misread the question and use $\text{A}\hat{\text{O}}\text{C} = 2.4$. If working shown, award ***M0*** then ***A0MRA1*** for the answer 48. Do not then penalize $\text{A}\hat{\text{O}}\text{C}$ in part (d) which, if used, leads to the answer $679.498\ldots$

However, if they use the prematurely rounded value of 2.4 for $\text{A}\hat{\text{O}}\text{C}$, penalise 1 mark for premature rounding for the answer 48 in (c). Do not then penalize for this in (d).

[3 marks]

1b. Find the length of the chord [AB]. *[3 marks]*

Markscheme

Note: In this question, do not penalise for missing or incorrect units. They are not included in the markscheme, to avoid complex answer lines.

METHOD 1

choosing cosine rule (must have cos in it) ***(M1)***

e.g. $c^2 = a^2 + b^2 - 2ab\cos C$

correct substitution (into rhs) ***A1***

e.g. $20^2 + 20^2 - 2(20)(20)\cos 1.5$, $\text{AB} = \sqrt{800 - 800\cos 1.5}$

$\text{AB} = 27.26555\ldots$

$\text{AB} = 27.3$ $[27.2, 27.3]$ ***A1 N2***

[3 marks]

METHOD 2

choosing sine rule ***(M1)***

e.g. $\frac{\sin A}{a} = \frac{\sin B}{b}$, $\frac{\text{AB}}{\sin O} = \frac{\text{AO}}{\sin B}$

correct substitution ***A1***

e.g. $\frac{\text{AB}}{\sin 1.5} = \frac{20}{\sin(0.5(\pi - 1.5))}$

$\text{AB} = 27.26555\ldots$

$\text{AB} = 27.3$ $[27.2, 27.3]$ ***A1 N2***

[3 marks]

1c. Find the area of triangle AOB. *[2 marks]*

Markscheme

correct substitution into area formula ***A1***

e.g. $\frac{1}{2}(20)(20)\sin 1.5$, $\frac{1}{2}(20)(27.2655504\ldots)\sin(0.5(\pi - 1.5))$

$\text{area} = 199.498997\ldots$ (accept $199.75106 = 200$, from using 27.3)

$\text{area} = 199$ $[199, 200]$ ***A1 N1***

[2 marks]

1d. Angle BOC is 2.4 radians. *[3 marks]*

Find the area of the shaded region.

Markscheme

calculating sector area using **their** angle AOC ***(A1)***

e.g. $\frac{1}{2}(2.38\ldots)(20^2)$, $200(2.38\ldots)$, $476.6370614\ldots$

shaded area **=** **their** area of triangle AOB **+** **their** area of sector ***(M1)***

e.g. $199.4989973\ldots + 476.6370614\ldots$, $199 + 476.637$

$\text{shaded area} = 676.136\ldots$ (accept $675.637\ldots = 676$ from using 199)

$\text{shaded area} = 676$ $[676, 677]$ ***A1 N2***

[3 marks]

1e. Angle BOC is 2.4 radians. *[4 marks]*

The shaded region is to be painted red. Red paint is sold in cans which cost \$32 each. One can covers 140 m^2 . How much does it cost to buy the paint?

Markscheme

dividing to find number of cans ***(M1)***

e.g. $\frac{676}{140}$, $4.82857\dots$

5 cans must be purchased ***(A1)***

multiplying to find cost of cans ***(M1)***

e.g. $5(32)$, $\frac{676}{140} \times 32$

cost is 160 (dollars) ***A1*** ***N3***

[4 marks]

Let $f(x) = \cos 2x$ and $g(x) = 2x^2 - 1$.

2a. Find $f\left(\frac{\pi}{2}\right)$. *[2 marks]*

Markscheme

$f\left(\frac{\pi}{2}\right) = \cos \pi$ ***(A1)***

$= -1$ ***A1*** ***N2***

[2 marks]

Let $f(x) = \cos 2x$ and $g(x) = 2x^2 - 1$.

2b. Find $(g \circ f)\left(\frac{\pi}{2}\right)$. *[2 marks]*

Markscheme

$(g \circ f)\left(\frac{\pi}{2}\right) = g(-1) \quad (= 2(-1)^2 - 1)$ ***(A1)***

$= 1$ ***A1*** ***N2***

[2 marks]

2c. Given that $(g \circ f)(x)$ can be written as $\cos(kx)$, find the value of k, $k \in \mathbb{Z}$. *[3 marks]*

Markscheme

$(g \circ f)(x) = 2(\cos(2x))^2 - 1 \quad (= 2\cos^2(2x) - 1)$ ***A1***

evidence of $2\cos^2\theta - 1 = \cos 2\theta$ (seen anywhere) ***(M1)***

$(g \circ f)(x) = \cos 4x$

$k = 4$ ***A1*** ***N2***

[3 marks]

Let $f(x) = (\sin x + \cos x)^2$.

3. Show that $f(x)$ can be expressed as $1 + \sin 2x$. *[2 marks]*

Markscheme

attempt to expand *(M1)*

e.g. $(\sin x + \cos x)(\sin x + \cos x)$; at least 3 terms

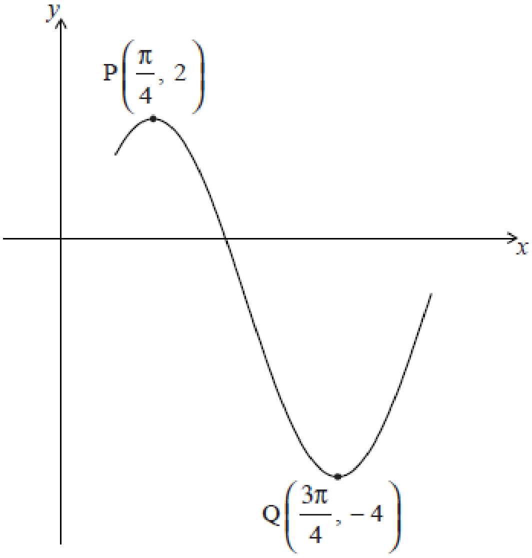
correct expansion ***A1***

e.g. $\sin^2 x + 2\sin x \cos x + \cos^2 x$

$f(x) = 1 + \sin 2x$ ***AG N0***

[2 marks]

The diagram below shows part of the graph of $f(x) = a\cos(b(x - c)) - 1$, where $a > 0$.



The point P $\left(\frac{\pi}{4}, 2\right)$ is a maximum point and the point Q $\left(\frac{3\pi}{4}, -4\right)$ is a minimum point.

4a. Find the value of a . *[2 marks]*

Markscheme

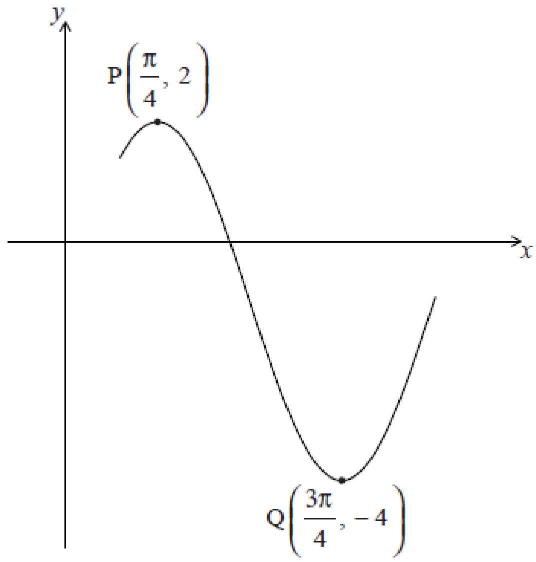
evidence of valid approach *(M1)*

e.g. $\frac{\max y \text{ value} - \min y \text{ value}}{2}$, distance from $y = -1$

$a = 3$ ***A1 N2***

[2 marks]

The diagram below shows part of the graph of $f(x) = a\cos(b(x - c)) - 1$, where $a > 0$.



The point P $\left(\frac{\pi}{4}, 2\right)$ is a maximum point and the point Q $\left(\frac{3\pi}{4}, -4\right)$ is a minimum point.

4b. (i) Show that the period of f is π . *[4 marks]*

(ii) Hence, find the value of b .

Markscheme

(i) evidence of valid approach ***(M1)***

e.g. finding difference in x-coordinates, $\frac{\pi}{2}$

evidence of doubling ***A1***

e.g. $2 \times \left(\frac{\pi}{2}\right)$

period $= \pi$ ***AG*** ***N0***

(ii) evidence of valid approach ***(M1)***

e.g. $b = \frac{2\pi}{\pi}$

$b = 2$ ***A1*** ***N2***

[4 marks]

4c. Given that $0 < c < \pi$, write down the value of c . *[1 mark]*

Markscheme

$c = \frac{\pi}{4}$ ***A1*** ***N1***

[1 mark]

Let $f(x) = \frac{3x}{2} + 1$, $g(x) = 4\cos\left(\frac{x}{3}\right) - 1$. Let $h(x) = (g \circ f)(x)$.

5a. Find an expression for $h(x)$. *[3 marks]*

Markscheme

attempt to form any composition (even if order is reversed) ***(M1)***

correct composition $h(x) = g\left(\frac{3x}{2} + 1\right)$ ***(A1)***

$h(x) = 4\cos\left(\frac{\frac{3x}{2}+1}{3}\right) - 1 \quad \left(4\cos\left(\frac{1}{2}x + \frac{1}{3}\right) - 1, 4\cos\left(\frac{3x+2}{6}\right) - 1\right)$ ***A1 N3***

[3 marks]

Let $f(x) = \frac{3x}{2} + 1$, $g(x) = 4\cos\left(\frac{x}{3}\right) - 1$. Let $h(x) = (g \circ f)(x)$.

5b. Write down the period of h . *[1 mark]*

Markscheme

period is $4\pi (12.6)$ ***A1 N1***

[1 mark]

5c. Write down the range of h . *[2 marks]*

Markscheme

range is $-5 \le h(x) \le 3 \quad ([-5, 3])$ ***A1A1 N2***

[2 marks]

6a. Show that $4 - \cos 2\theta + 5\sin\theta = 2\sin^2\theta + 5\sin\theta + 3$. *[2 marks]*

Markscheme

attempt to substitute $1 - 2\sin^2\theta$ for $\cos 2\theta$ ***(M1)***

correct substitution ***A1***

e.g. $4 - (1 - 2\sin^2\theta) + 5\sin\theta$

$4 - \cos 2\theta + 5\sin\theta = 2\sin^2\theta + 5\sin\theta + 3$ ***AG N0***

[2 marks]

6b. **Hence**, solve the equation $4 - \cos 2\theta + 5\sin\theta = 0$ for $0 \le \theta \le 2\pi$. *[5 marks]*

Markscheme

evidence of appropriate approach to solve ***(M1)***

e.g. factorizing, quadratic formula

correct working ***A1***

e.g. $(2\sin\theta + 3)(\sin\theta + 1)$, $(2x + 3)(x + 1) = 0$, $\sin x = \frac{-5 \pm \sqrt{1}}{4}$

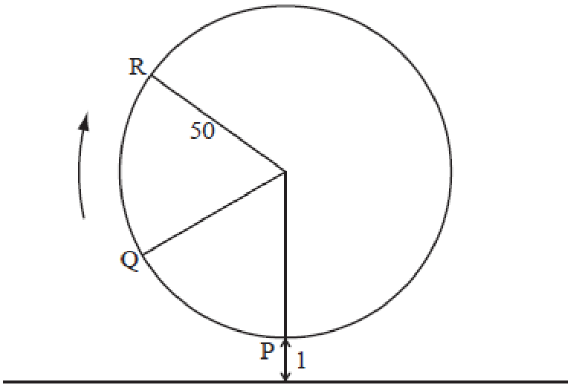
correct solution $\sin\theta = -1$ (do not penalise for including $\sin\theta = -\frac{3}{2}$ ***(A1)***

$\theta = \frac{3\pi}{2}$ ***A2 N3***

[5 marks]

The following diagram represents a large Ferris wheel at an amusement park.

The points P, Q and R represent different positions of a seat on the wheel.



The wheel has a radius of 50 metres and rotates clockwise at a rate of one revolution every 30 minutes.

A seat starts at the lowest point P, when its height is one metre above the ground.

7a. Find the height of a seat above the ground after 15 minutes. *[2 marks]*

Markscheme

valid approach *(M1)*

e.g. 15 mins is half way, top of the wheel, $d + 1$

height $= 101$ (metres) ***A1*** ***N2***

[2 marks]

7b. After six minutes, the seat is at point Q. Find its height above the ground at Q. *[5 marks]*

Markscheme

evidence of identifying rotation angle after 6 minutes ***A1***

e.g. $\frac{2\pi}{5}$, $\frac{1}{5}$ of a rotation, 72°

evidence of appropriate approach *(M1)*

e.g. drawing a right triangle and using cosine ratio

correct working (seen anywhere) ***A1***

e.g. $\cos \frac{2\pi}{5} = \frac{x}{50}$, $15.4(508\ldots)$

evidence of appropriate method ***M1***

e.g. height $=$ radius $+ 1 - 15.45\ldots$

height $= 35.5$ (metres) (accept 35.6) ***A1*** ***N2***

[5 marks]

7c. The height of the seat above ground after t minutes can be modelled by the function $h(t) = 50\sin(b(t - c)) + 51$. *[6 marks]*

Find the value of b and of c .

Markscheme

METHOD 1

evidence of substituting into $b = \frac{2\pi}{\text{period}}$ ***(M1)***

correct substitution

e.g. period = 30 minutes, $b = \frac{2\pi}{30}$ ***A1***

$b = 0.209 \ \left(\frac{\pi}{15}\right)$ ***A1 N2***

substituting into $h(t)$ ***(M1)***

e.g. $h(0) = 1$, $h(15) = 101$

correct substitution ***A1***

$1 = 50\sin\left(-\frac{\pi}{15}c\right) + 51$

$c = 7.5$ ***A1 N2***

METHOD 2

evidence of setting up a system of equations ***(M1)***

two correct equations

e.g. $1 = 50\sin b(0 - c) + 51$, $101 = 50\sin b(15 - c) + 51$ ***A1A1***

attempt to solve simultaneously ***(M1)***

e.g. evidence of combining two equations

$b = 0.209 \ \left(\frac{\pi}{15}\right)$, $c = 7.5$ ***A1A1 N2N2***

[6 marks]

7d. The height of the seat above ground after *t* minutes can be modelled by the function $h(t) = 50\sin(b(t - c)) + 51$. *[3 marks]*

Hence find the value of *t* the first time the seat is 96 m above the ground.

Markscheme

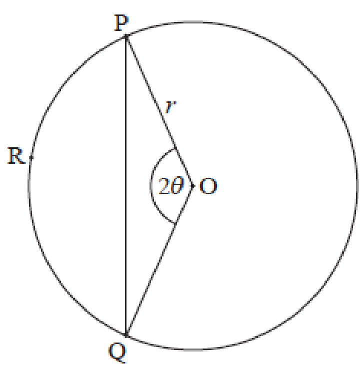
evidence of solving $h(t) = 96$ ***(M1)***

e.g. equation, graph

$t = 12.8$ (minutes) ***A2 N3***

[3 marks]

Consider the following circle with centre O and radius *r* .



The points P, R and Q are on the circumference, $\mathrm{P\hat{O}Q} = 2\theta$, for $0 < \theta < \frac{\pi}{2}$.

8a. Use the cosine rule to show that $\mathrm{PQ} = 2r\sin\theta$. *[4 marks]*

Markscheme

correct substitution into cosine rule ***A1***

e.g. $\text{PQ}^2 = r^2 + r^2 - 2(r)(r)\cos(2\theta)$, $\text{PQ}^2 = 2r^2 - 2r^2(\cos(2\theta))$

substituting $1 - 2\sin^2\theta$ for $\cos 2\theta$ (seen anywhere) ***A1***

e.g. $\text{PQ}^2 = 2r^2 - 2r^2(1 - 2\sin^2\theta)$

working towards answer ***(A1)***

e.g. $\text{PQ}^2 = 2r^2 - 2r^2 + 4r^2\sin^2\theta$

recognizing $2r^2 - 2r^2 = 0$ (including crossing out) (seen anywhere)

e.g. $\text{PQ}^2 = 4r^2\sin^2\theta$, $\text{PQ} = \sqrt{4r^2\sin^2\theta}$

$\text{PQ} = 2r\sin\theta$ ***AG*** ***N0***

[4 marks]

8b. Let l be the length of the arc PRQ . *[5 marks]*

Given that $1.3\text{PQ} - l = 0$, find the value of θ .

Markscheme

$\text{PRQ} = r \times 2\theta$ (seen anywhere) ***(A1)***

correct set up ***A1***

e.g. $1.3 \times 2r\sin\theta - r \times (2\theta) = 0$

attempt to eliminate r ***(M1)***

correct equation in terms of the one variable θ ***(A1)***

e.g. $1.3 \times 2\sin\theta - 2\theta = 0$

1.221496215

$\theta = 1.22$ (accept 70.0° (69.9)) ***A1*** ***N3***

[5 marks]

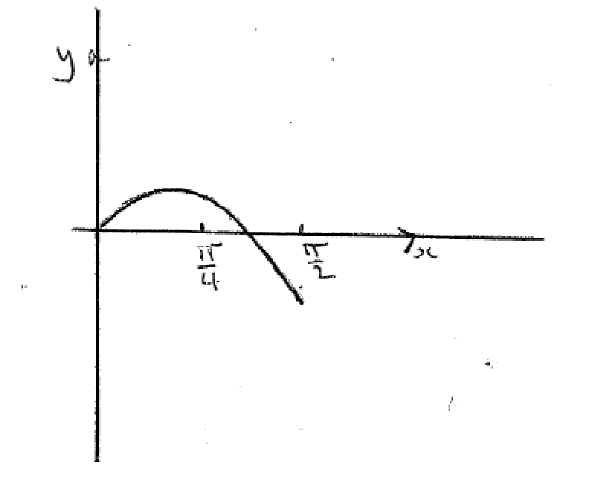
8c. Consider the function $f(\theta) = 2.6\sin\theta - 2\theta$, for $0 < \theta < \frac{\pi}{2}$. *[4 marks]*

(i) Sketch the graph of f .

(ii) Write down the root of $f(\theta) = 0$.

Markscheme

(i)



A1A1A1 *N3*

Note: Award ***A1*** for approximately correct shape, ***A1*** for x-intercept in approximately correct position, ***A1*** for domain. Do not penalise if sketch starts at origin.

(ii) 1.221496215

$\theta = 1.22$ ***A1*** ***N1***

[4 marks]

8d. Use the graph of f to find the values of θ for which $l < 1.3\text{PQ}$. *[3 marks]*

Markscheme

evidence of appropriate approach (may be seen earlier) ***M2***

e.g. $2\theta < 2.6\sin\theta$, $0 < f(\theta)$, showing positive part of sketch

$0 < \theta < 1.221496215$

$0 < \theta = 1.22$ (accept $\theta < 1.22$) ***A1*** ***N1***

[3 marks]

Consider the triangle ABC, where AB =10 , BC = 7 and $\text{C}\widehat{\text{A}}\text{B} = 30^\circ$.

9a. Find the two possible values of $\text{A}\widehat{\text{C}}\text{B}$. *[4 marks]*

Markscheme

Note: accept answers given in degrees, and minutes.

evidence of choosing sine rule ***(M1)***

e.g. $\frac{\sin A}{a} = \frac{\sin B}{b}$

correct substitution ***A1***

e.g. $\frac{\sin\theta}{10} = \frac{\sin 30^\circ}{7}$, $\sin\theta = \frac{5}{7}$

$\text{A}\widehat{\text{C}}\text{B} = 45.6^\circ$, $\text{A}\widehat{\text{C}}\text{B} = 134^\circ$ ***A1A1*** ***N1N1***

Note: If candidates only find the acute angle in part (a), award no marks for (b).

[4 marks]

9b. Hence, find $\mathrm{A\widehat{B}C}$, given that it is acute. *[2 marks]*

Markscheme

attempt to substitute their larger value into angle sum of triangle ***(M1)***

e.g. $180^\circ - (134.415\ldots^\circ + 30^\circ)$

$\mathrm{A\widehat{B}C} = 15.6^\circ$ ***A1*** ***N2***

[2 marks]