CHAPTER 8

Section 8–1

2. There are an infinite number of possible triangles, all similar, with three given angles whose sum is 180°.

4. If two angles α and β of a triangle are known, the third angle can be found immediately as $\gamma = 180° - \alpha - \beta$. Then the given side will be included between two of the three known angles.

6. It is called the ambiguous case because there are two triangles determined by the given information.

Note: Answers have been rounded to the number of significant digits given in Table 1; an = sign has been used rather than ≈.

8. $\gamma = 180° - (41° + 33°)$

$\gamma = 106°$

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma} \qquad \frac{a}{\sin 41°} = \frac{21}{\sin 106°} \qquad a = 14 \text{ cm}$$

$$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \qquad \frac{b}{\sin 33°} = \frac{21}{\sin 106°} \qquad b = 12 \text{ cm}$$

10. $\alpha = 180° - (43° + 36°)$

$\alpha = 101°$

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma} \qquad \frac{92}{\sin 101°} = \frac{c}{\sin 36°} \qquad c = 55 \text{ mm}$$

$$\frac{b}{\sin\beta} = \frac{a}{\sin\alpha} \qquad \frac{b}{\sin 43°} = \frac{92}{\sin 101°} \qquad b = 64 \text{ mm}$$

12. $\beta = 180° - (52° + 105°)$

$\beta = 23°$

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma} \qquad \frac{a}{\sin 52°} = \frac{47}{\sin 105°} \qquad a = 38 \text{ m}$$

$$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \qquad \frac{b}{\sin 23°} = \frac{47}{\sin 105°} \qquad b = 19 \text{ m}$$

14. $\alpha = 180° - (83° + 77°)$

$\alpha = 20°$

$$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \qquad \frac{b}{\sin 83°} = \frac{25}{\sin 77°} \qquad b = 25 \text{ mi}$$

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma} \qquad \frac{a}{\sin 20°} = \frac{25}{\sin 77°} \qquad a = 8.8 \text{ mi}$$

16. $a = 3$ ft, $b = 6$ ft, $\alpha = 30°$
SSA:
$h = b \sin\alpha = 6 \sin 30° = 3 = a \Rightarrow$ 1 triangle; case (b)

18. $a = 8$ ft, $b = 6$ ft, $\alpha = 30°$
SSA:
$h = b \sin\alpha = 6 \sin 30° = 3$
$8 > 3 \Rightarrow$ 1 triangle; case (d)

20. $a = 2$ ft, $b = 6$ ft, $\alpha = 30°$
SSA:
$h = b \sin\alpha = 6 \sin 30° = 3 > 2 \Rightarrow$ 0 triangles; case (a)

22. $a = 5$ ft, $b = 6$ ft, $\alpha = 30°$
SSA:
$h = b \sin\alpha = 6 \sin 30° = 3$
$3 < 5 < 6 \Rightarrow$ 2 triangles; case (c)

24. $\beta = 27.5°$, $\gamma = 54.5°$, $a = 9.27$ inches

$\alpha = 180° - (27.5° + 54.5°)$

$\alpha = 98°$

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$
$$\frac{9.27}{\sin 98°} = \frac{b}{\sin 27.5°}$$
$$b = 4.32 \text{ in}$$

$$\frac{c}{\sin\gamma} = \frac{a}{\sin\alpha}$$
$$\frac{c}{\sin 54.5°} = \frac{9.27}{\sin 98°}$$
$$c = 7.62 \text{ in}$$

26. $\alpha = 122.7°$, $\beta = 34.4°$, $b = 18.3$ km

$\gamma = 180° - (122.7° + 34.4°)$

$\gamma = 22.9°$

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$
$$\frac{a}{\sin 122.7°} = \frac{18.3}{\sin 34.4°}$$
$$a = 27.3 \text{ km}$$

$$\frac{c}{\sin\gamma} = \frac{b}{\sin\beta}$$
$$\frac{c}{\sin 22.9°} = \frac{18.3}{\sin 34.4°}$$
$$c = 12.6 \text{ km}$$

28. $\alpha = 26.3°$, $a = 14.7$ inches, $b = 35.2$ inches

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$
$$\frac{14.7}{\sin 26.3°} = \frac{35.2}{\sin\beta}$$
$$\sin\beta = 1.0609596 > 1 \Rightarrow \text{No solution}$$

30. $h = b \sin\alpha = 172 \sin 43.5° = 118.4$

$a = 138$; $h < a < b$, so two triangles possible

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$$
$$\frac{\sin 43.5°}{138} = \frac{\sin\beta}{172}$$
$$\sin\beta = 0.8579491813$$
$$\beta = 59.1° \text{ or } \beta = 120.9°$$

Triangle I

$\beta = 59.1°$

$\alpha + \beta + \gamma = 43.5° + 59.1° + \gamma = 180° \Rightarrow \gamma = 77.4°$

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c}$$
$$\frac{\sin 43.5°}{138} = \frac{\sin 77.4°}{c} \Rightarrow c = 196 \text{ cm}$$

Triangle II

$\beta' = 120.9°$

$\alpha + \beta' + \gamma' = 43.5° + 120.9° + \gamma' = 180° \Rightarrow \gamma' = 15.6°$

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma'}{c'}$$
$$\frac{\sin 43.5°}{138} = \frac{\sin 15.6°}{c} \Rightarrow c = 53.9 \text{ cm}$$

32. $h = a \sin\beta = 244 \sin 27.3° = 111.91$; $b = 135$

$h < b < a$, so two triangles possible

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$$
$$\frac{\sin\alpha}{244} = \frac{\sin 27.3°}{135}$$
$$\sin\alpha = 0.8289666022$$
$$\alpha = 56.0° \text{ or } \alpha = 124.0°$$

Triangle I

$\alpha = 56.0°$

$\alpha + \beta + \gamma = 56.0° + 27.3° + \gamma = 180° \Rightarrow \gamma = 96.7°$

$$\frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$
$$\frac{\sin 27.3°}{135} = \frac{\sin 96.7°}{c} \Rightarrow c = 292 \text{ cm}$$

Triangle II

$\alpha' = 124.0°$

$\alpha' + \beta + \gamma' = 124.0° + 27.3° + \gamma' = 180° \Rightarrow \gamma' = 28.7°$

$$\frac{\sin\beta}{b} = \frac{\sin\gamma'}{c'}$$
$$\frac{\sin 27.3°}{135} = \frac{\sin 28.7°}{c'} \Rightarrow c' = 141 \text{ cm}$$

34. $\alpha = 137.3°$, $a = 13.9$ m, $b = 19.1$ m

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

$$\frac{13.9}{\sin 137.3°} = \frac{19.1}{\sin\beta} \Rightarrow \beta = 68.73°$$

$\beta + \alpha = 206.03° > 180°$

No solution

36. $\beta = 33°50'$, $a = 673$ m, $b = 1240$ m

$$\frac{b}{\sin\beta} = \frac{a}{\sin\alpha} \qquad \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

$$\frac{1240}{\sin 33°50'} = \frac{673}{\sin\alpha} \qquad \frac{1240}{\sin 33°50'} = \frac{c}{\sin 128°30'}$$

$\alpha = 17°40'$ $\qquad c = 1740$ m

$\gamma = 180° - (33°50' + 17°40')$

$\gamma = 128°30'$

38. $h = a\sin\beta = 92\sin 30° = 46$

$h = b$, so there is one triangle

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$$

$$\frac{\sin\alpha}{92} = \frac{\sin 30°}{46}$$

$\sin\alpha = 1 \Rightarrow \alpha = 90°$

$\alpha + \beta + \gamma = 90° + 30° + \gamma = 180° \Rightarrow \gamma = 60°$

$$\frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

$$\frac{\sin 30°}{46} = \frac{\sin 60°}{c} \Rightarrow c = 80 \text{ in}$$

40. $\alpha = 37.3°$, $b = 42.8$ cm

$k = 42.8\sin 37.3° = 25.9$ is k such that $0 < a < k$ gives no solution; $a = k$ gives one solution; $k < a < b$ gives two solutions.

42. From the law of sines $\dfrac{a}{\sin\alpha} = \dfrac{c}{\sin\gamma} \Rightarrow \dfrac{a}{c} = \dfrac{\sin\alpha}{\sin\gamma}$ (1)

and $\dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma} \Rightarrow \dfrac{b}{c} = \dfrac{\sin\beta}{\sin\gamma}$ (2)

$$\frac{a+b}{c} = \frac{\sin\alpha + \sin\beta}{\sin\gamma}$$ adding (1) and (2)

$$= \frac{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}}$$ but $\sin\dfrac{\alpha+\beta}{2} = \sin\dfrac{1}{2}(180 - \gamma) = \sin\left(90 - \dfrac{\gamma}{2}\right) = \cos\dfrac{\gamma}{2}$

$$= \frac{\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{\sin\frac{\gamma}{2}\sin\frac{\alpha+\beta}{2}}$$

$$\frac{a+b}{c} = \frac{\cos\frac{\alpha-\beta}{2}}{\sin\frac{\gamma}{2}} \quad (3)$$

similarly, subtracting (1) and (2)

$$\frac{a-b}{c} = \frac{\sin\alpha - \sin\beta}{\sin\gamma} = \frac{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}{\sin\gamma}$$

$$= \frac{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}{2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}}$$ but $\cos\dfrac{\alpha+\beta}{2} = \cos\dfrac{1}{2}(180 - \gamma) = \cos\left(90 - \dfrac{\gamma}{2}\right) = \sin\dfrac{\gamma}{2}$

$$= \frac{\sin\frac{\gamma}{2}\sin\frac{\alpha-\beta}{2}}{\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}}$$

$$\frac{a-b}{c} = \frac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\gamma}{2}} \quad (4)$$ dividing (4) by (3)

$$\frac{\frac{a-b}{c}}{\frac{a+b}{c}} = \frac{\frac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\gamma}{2}}}{\frac{\cos\frac{\alpha-\beta}{2}}{\sin\frac{\gamma}{2}}}$$

$$\frac{a-b}{c}\cdot\frac{c}{a+b} = \frac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\gamma}{2}}\cdot\frac{\sin\frac{\gamma}{2}}{\cos\frac{\alpha-\beta}{2}}$$

$$\frac{a-b}{a+b} = \tan\frac{\alpha-\beta}{2}\ \tan\frac{\gamma}{2} \text{ but } \tan\frac{\gamma}{2} = \cot\frac{\alpha+\beta}{2}$$

$$= \tan\frac{\alpha-\beta}{2}\cdot\frac{1}{\tan\frac{\alpha+\beta}{2}}$$

$$\frac{a-b}{a+b} = \frac{\tan\frac{\alpha-\beta}{2}}{\tan\frac{\alpha+\beta}{2}}$$

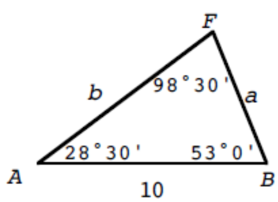
(B) from 7, $a = 41$ $\alpha = 73°$ $b = 20$ $\beta = 28°$

$$\frac{\tan\frac{73°-28°}{2}}{\tan\frac{73°+28°}{2}} = 0.3415$$

$$\frac{41-20}{41+20} = 0.3443$$

Answers agree to two significant digits, the accuracy of the given information.

44.

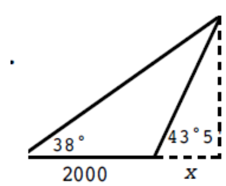


$$\frac{b}{\sin 53°} = \frac{a}{\sin 28°30'} = \frac{10}{\sin 98°30'}$$

$b = 8.08$ miles from A

$a = 4.82$ miles from B

46.



$$\tan 43°5' = \frac{h}{x}$$

$$x = \frac{h}{\tan 43°5'}$$

$$\tan 38° = \frac{h}{2000+x} = \frac{h}{2000+\frac{h}{\tan 43°5'}}$$

$$2000\tan 38° + h\left(\frac{\tan 38°}{\tan 43°5'}\right) = h$$

$$h\left(1-\frac{\tan 38°}{\tan 43°5'}\right) = 2000\tan 38°$$

$$h = \frac{2000\tan 38°}{1-\frac{\tan 38°}{\tan 43°5'}}$$

$$h \approx 9492.39$$

The distance above sea level = 5000 + h = 14,490 feet to 4 significant digits.

48. $\dfrac{6.3}{\sin\alpha} = \dfrac{1.7}{\sin 11°} = \dfrac{c}{\sin\gamma}$

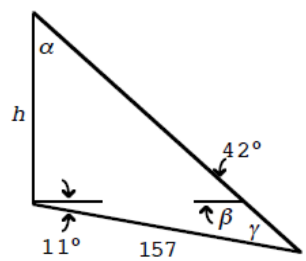
$\alpha = 45° \Rightarrow \gamma = 124° \Rightarrow c = 7.4"$

$\alpha = 135° \Rightarrow \gamma = 34° \Rightarrow c = 5.0"$

50. $\sin(SEV) = \dfrac{1.085\times10^8}{1.495\times10^8}$

$SEV = 46.5°$

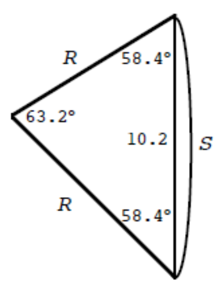
52.



$$\alpha + 42° = 90°$$
$$\alpha = 48°$$
$$\beta + 42° = 180°$$
$$\beta = 138°$$

$$11° + \beta + \gamma = 180°$$
$$11° + 138° + \gamma = 180°$$
$$\gamma = 31°$$
$$\frac{h}{\sin\gamma} = \frac{157}{\sin\alpha}$$
$$\frac{h}{\sin 31°} = \frac{157}{\sin 48°}$$
$h = 109$ ft, to the nearest foot

54.



$$\frac{180° - 63.2°}{2} = 58.4°$$
$$\frac{10.2}{\sin 63.2} = \frac{R}{\sin 58.4°}$$
$R = 9.73$ mm to 3 significant digits

$s = R \cdot 63.2° \cdot \dfrac{\pi}{180°} = 10.7$ mm to 3 significant digits

56. Let x be the length of the side in the horizontal triangle that is also in the vertical triangle with angle γ.

In the horizontal triangle

$$\frac{x}{\sin\alpha} = \frac{d}{\sin(180° - (\alpha + \beta))}$$
$$x = \text{d} \sin\alpha \csc(\alpha + \beta)$$

In the vertical triangle

$$\tan\gamma = \frac{h}{x}$$
$$h = x \tan\gamma = d \sin\alpha \csc(\alpha + \beta) \tan\gamma$$

$$\sin(180° - (\alpha + \beta)) = \sin 180° \cos(\alpha + \beta) - \cos 180° \sin(\alpha + \beta)$$
$$= 0 \cdot \cos(\alpha + \beta) - (-1) \sin(\alpha + \beta)$$
$$= \sin(\alpha + \beta)$$
$$= \frac{1}{\csc(\alpha + \beta)}$$

Section 8–2

2. Use the law of cosines to determine the largest angle (largest because it is opposite the longest side). Then use the law of sines to determine a second angle and the fact that the sum of the angles is 180° to determine the third.

4. Substituting $a = b = c$ into $c^2 = a^2 + b^2 - 2ab \cos\gamma$ gives $c^2 = c^2 + c^2 - 2c^2 \cos\gamma$ which reduces to $\cos\gamma = \frac{1}{2}$ or $\gamma = 60°$, confirming that in an equilateral triangle all angles measure 60°.

6. There is no SSA congruence theorem, since the given information does not determine a triangle uniquely. There is an SSS congruence theorem since three sides that satisfy the triangle inequality determine the triangle uniquely.

8. A triangle can have at most one obtuse angle. Since $\alpha = 93.5°$ is obtuse both γ and β must be acute. [$\beta + \gamma = 180° - 93.5° = 86.5°$; thus both β and γ are less than 90°.]

10. $\beta = 57.3°$, $a = 6.08$ cm, $c = 5.25$ cm
$a^2 + c^2 - 2ac\cos\beta = b^2$
$6.08^2 + 5.25^2 - 2(6.08)(5.25)\cos 57.3° = b^2 \Rightarrow b = 5.48$ cm
Solve for smallest angle:

$$\frac{5.25}{\sin\gamma} = \frac{5.48}{\sin 57.3°} \qquad \frac{6.08}{\sin\alpha} = \frac{5.48}{\sin 57.3°}$$

$$\gamma = 53.7° \qquad \alpha = 69.0° \text{ or } \alpha = 180° - (57.3° + 53.7°)$$

12. $\alpha = 135°50'$, $b = 8.44$ in, $c = 20.3$ in
$b^2 + c^2 - 2bc\cos\alpha = a^2$
$8.44^2 + 20.3^2 - 2(8.44)(20.3)\cos 135°50' = a^2 \Rightarrow a = 27.0$ in
Solve for smallest angle:

$$\frac{27}{\sin 135°50'} = \frac{8.44}{\sin\beta} \qquad \frac{20.3}{\sin\gamma} = \frac{27}{\sin 135°50'}$$

$$\beta = 12°30' \qquad \gamma = 31°40' \text{ or } \gamma = 180° - (135°50' + 12°30')$$

14. If $a = 12.5$ cm, $b = 25.3$ cm, $c = 10.7$ cm, then sides a and c are not long enough to construct a triangle:
$a + c < b$

16. $a = 10.5$ mi, $b = 20.7$ mi, $c = 12.2$ mi
Solve for the largest angle:

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$
$$20.7^2 = 10.5^2 + 12.2^2 - 2(10.5)(12.2)\cos\beta$$
$$\cos\beta \approx -0.6612021858$$
$$\beta = 131.4°$$
$$\frac{20.7}{\sin 131.4} = \frac{12.2}{\sin\gamma} \Rightarrow \gamma = 26.2°$$
$$\alpha = 180° - (131.4° + 26.2°) = 22.4°$$

18. $a = 31.5$ m, $b = 29.4$ m, $c = 33.7$ m
Solving for the largest angle:

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$
$$33.7^2 = 31.5^2 + 29.4^2 - 2(31.5)(29.4)\cos\gamma$$
$$\cos\gamma \approx 0.389223626$$
$$\gamma = 67.1°$$
$$\frac{33.7}{\sin 67.1°} = \frac{31.5}{\sin\alpha} \Rightarrow \alpha = 59.4°$$
$$\beta = 180° - (67.1° + 59.4°) = 53.5°$$

20. $\beta + \gamma = 85.6° + 97.3° = 182.9° > 180° \Rightarrow$ no solution

22. $\beta = 27.3°$, $a = 13.7$ yds, $c = 20.1$ yd

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$
$$= 13.7^2 + 20.1^2 - 2(13.7)(20.1)\cos 27.3°$$
$$b = 10.1 \text{ yd}$$
$$\frac{10.1}{\sin 27.3°} = \frac{13.7}{\sin\alpha}$$
$$\alpha = 38.5°$$
$$\gamma = 180° - (27.3° + 38.5°) = 114.2°$$

(Answers will vary slightly depending on method used.)

24. $\beta = 132.4°$, $\gamma = 17.3°$, $b = 67.6$ km
$\alpha = 180° - (\beta + \gamma) = 180° - (132.4° + 17.3°) = 30.3°$

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$
$$\frac{a}{\sin 30.3°} = \frac{67.6}{\sin 132.4°} = \frac{c}{\sin 17.3°}$$
$$a = 46.2 \text{ km}; c = 27.2 \text{ km}$$

26. $\gamma = 66.4°$, $b = 25.5$ m, $c = 25.5$ m
$b = c \Rightarrow \gamma = \beta = 66.4°$
$\alpha = 180° - (\gamma + \beta) = 180° - (2(66.4°)) = 47.2°$

$$\frac{25.5}{\sin 66.4°} = \frac{a}{\sin 47.2°}$$
$$a = 20.4 \text{ m}$$

28. $a = 10.5$ cm, $b = 5.23$ cm, $c = 8.66$ cm

30. $\alpha = 46.7°$, $a = 18.1$ m, $b = 22.6$ m

Angle opposite longest side:

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$10.5^2 = 5.23^2 + 8.66^2 - 2(5.23)(8.66)\cos\alpha$$
$$\alpha = 95.0°$$
$$\frac{10.5}{\sin 95.0°} = \frac{5.23}{\sin\beta}$$
$$\beta = 29.7°$$
$$\gamma = 180° - (\alpha + \beta) = 180° - (95° + 29.7°)$$
$$\gamma = 55.3°$$

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$
$$\frac{18.1}{\sin 46.7} = \frac{22.6}{\sin\beta} \Rightarrow \beta = 65.3° \text{ or } \beta' = 114.7°$$
$$\beta = 65.3° \Rightarrow \gamma = 180° - (65.3° + 46.7°) = 68.0°$$
$$\beta' = 114.7° \Rightarrow \gamma' = 180° - (114.7° + 46.7°) = 18.6°$$
$$\frac{18.1}{\sin 46.7°} = \frac{c}{\sin 68.0°} = \frac{c'}{\sin 18.6°}$$
$$c = 23.1 \text{ m}$$
$$c' = 7.93 \text{ m}$$

Triangle I: $\beta = 65.3°$, $\gamma = 68.0°$, $c = 23.1$ m

Triangle II: $\beta' = 114.7°$, $\gamma' = 18.6°$, $c' = 7.93$ m

32. $\gamma = 47.9°$, $b = 35.2$ in, $c = 25.5$ in

$$\frac{c}{\sin\gamma} = \frac{b}{\sin\beta}$$
$$\frac{25.5}{\sin 47.9°} = \frac{35.2}{\sin\beta}$$
$$\sin\beta = 1.0242176 \Rightarrow \text{No solution}$$

34. $h = c\sin\beta = 98.5\sin 25.1° = 41.8$

$h < b < c \Rightarrow$ There are two triangles.

$$\frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$
$$\frac{\sin 25.1°}{53.7} = \frac{\sin\gamma}{98.5}$$
$$\sin\gamma = 0.7780939132$$
$$\gamma = 51.1° \text{ or } 128.9°$$

Triangle I

$\gamma = 51.1°$

$\alpha + \beta + \gamma = \alpha + 25.1° + 51.1° = 180° \Rightarrow \alpha = 103.8°$

$$\frac{\sin\beta}{b} = \frac{\sin\alpha}{a}$$
$$\frac{\sin 25.1°}{53.7} = \frac{\sin 103.8°}{a}$$
$$a = 123 \text{ m}$$

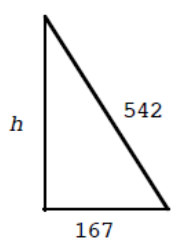
Triangle II

$\gamma' = 128.9°$

$\alpha' + \beta + \gamma' = \alpha' + 25.1° + 128.9° = 180° \Rightarrow \alpha' = 26.0°$

$$\frac{\sin\beta}{b} = \frac{\sin\alpha'}{a'}$$
$$\frac{\sin 25.1°}{53.7} = \frac{\sin 26.0°}{a'}$$
$$a' = 55.5 \text{ m}$$

36.



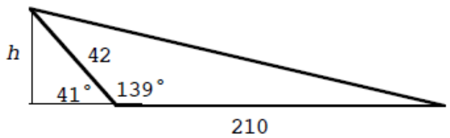
$$h = \sqrt{542^2 - 167^2}$$
$$A = \frac{1}{2}(167)\left(\sqrt{542^2 - 167^2}\right) = 43{,}100 \text{ sq. yds.}$$

38.

$$\sin 49.3° = \frac{h}{10.1} \qquad h = 10.1\sin 49.3°$$
$$A = \frac{1}{2}(183)\left(10.1\sin 49.3°\right) = 701 \text{ sq. m.}$$

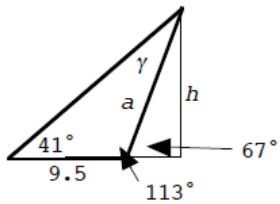
40.

$$\sin 41° = \frac{h}{42} \qquad h = 42\sin 41°$$



$$A = \frac{1}{2}(210)\left(42\sin 41^{\circ}\right) = 2{,}900 \text{ sq. ft.}$$

42.



$$\gamma = 180^{\circ} - \left(41^{\circ} + 113^{\circ}\right) = 26^{\circ}$$

$$\frac{\sin 26^{\circ}}{9.5} = \frac{\sin 41^{\circ}}{a} \qquad a = \frac{9.5\sin 41^{\circ}}{\sin 26^{\circ}}$$

$$\sin 67^{\circ} = \frac{h}{\dfrac{9.5\sin 41^{\circ}}{\sin 26^{\circ}}} \qquad h = \frac{9.5\sin 41^{\circ}}{\sin 26^{\circ}}\cdot\sin 67^{\circ}$$

$$A = \frac{1}{2}(9.5)\left(\frac{9.5\sin 41^{\circ}}{\sin 26^{\circ}}\cdot\sin 67^{\circ}\right) = 62 \text{ sq. m.}$$

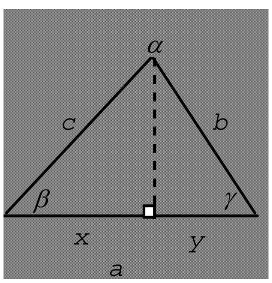
44. $s = \dfrac{95+19+104}{2} = 109$

$$A = \sqrt{109(109-95)(109-19)(109-104)} = 830 \text{ sq. yds.}$$

46. Given $c^2 = a^2 + b^2$:

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos\gamma && \text{law of cosines} \\ a^2 + b^2 &= a^2 + b^2 - 2ab\cos\gamma && \text{substitution} \\ -2ab\cos\gamma &= 0 \\ \cos\gamma &= 0 \\ \gamma &= 90^{\circ} \end{aligned}$$

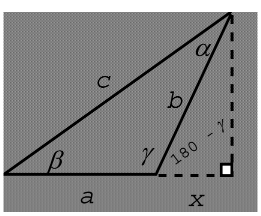
48.



$$a = x + y : \cos\beta = \frac{x}{c} \Rightarrow x = c\cos\beta$$

$$: \cos\gamma = \frac{y}{b} \Rightarrow y = b\cos\gamma$$

$$a = c\cos\beta + b\cos\gamma$$



$$\begin{aligned} \cos\beta &= \frac{a+x}{c} \\ a + x &= c\cos\beta \\ a &= c\cos\beta - x \\ a &= c\cos\beta - (-b\cos\gamma) \\ a &= c\cos\beta + b\cos\gamma \end{aligned}$$

$$\begin{aligned} \cos(180-\gamma) &= \frac{x}{b} \\ -\cos\gamma &= \frac{x}{b} \\ x &= -b\cos\gamma \end{aligned}$$

50. It is wiser to start by using the law of cosines to find the largest angle, in this case β. Then

$$\cos\beta = \frac{a^2+c^2-b^2}{2ac} = \frac{1^2+1^2-\left(\sqrt{3}\right)^2}{2\cdot 1\cdot 1} = -\frac{1}{2}$$

so $\beta = 120^{\circ}$. Then since $a = c$, α must equal γ, hence from $\alpha + \gamma = 180^{\circ} - 120^{\circ}$ we get $\alpha = \gamma = 30^{\circ}$. By finding α first, the incorrect conclusion $\beta = 30^{\circ}$ was drawn from the equation $\sin\beta = \dfrac{\sqrt{3}}{2}$; actually, as we just saw, $\beta = 120^{\circ}$.

52. $AB = \sqrt{85^2 + 73^2 - 2(85)(73)\cos 110^{\circ}} \approx 130$ m

54. $13.8^2 = 8.26^2 + 8.26^2 - 2(8.26)(8.26)\cos\theta$

56. $c = \sqrt{8^2 + 3^2 - 2(8)(3)\cos(144^{\circ}50')} \approx 10.6$ ft

$\theta \approx 113.3°$

58. After 2 hours, Plane A has traveled 800 miles, Plane B 1000 miles. The angle between them is 45°.
$c = \sqrt{1000^2 + 800^2 - 2(1000)(800)\cos 45°} \approx 713$ mi

60. The angle at the center is $\frac{360°}{9} = 40°$. An isosceles triangle is formed, so the other two angles are $70°\left(\frac{180° - 40°}{2}\right)$. Let x be the chord formed in the circle. Using the law of sines to solve,

$$\frac{7.09}{\sin 70°} = \frac{x}{\sin 40°}$$

$x \approx 4.849846$ The perimeter, to 3 significant digits, is $9x = 43.6$ cm.

62.
$OA = \sqrt{3^2 + 4^2} = 5$
$OB = \sqrt{5^2 + 1^2} = \sqrt{26}$
$AB = \sqrt{(4-5)^2 + (3-1)^2} = \sqrt{5}$

$AB^2 = OA^2 + OB^2 - 2(OA)(OB)\cos\theta$
$(\sqrt{5})^2 = 5^2 + (\sqrt{26})^2 - 2(5)(\sqrt{26})\cos\theta$
$\theta \approx 0.446$ radian

64. The sides of the triangle have lengths of 5 + 2 = 7, 8 + 2 = 10, and 8 + 5 = 13. Find angle γ first (angle opposite longest side):
$13^2 = 10^2 + 7^2 - 2(10)(7)\cos\gamma$
$\gamma \approx 98.2132 \approx 98°10'$
Angle α: $\frac{13}{\sin 98°10} = \frac{7}{\sin\alpha}$
$\alpha \approx 32.20845 \approx 32°10'$
Angle β: $\beta = 180° - (\alpha + \gamma) \approx 180° - (98°10' + 32°10') \approx 49°40'$

66.
$AB = \sqrt{4.3^2 + 8.1^2} = \sqrt{84.1}$
$AC = \sqrt{8.1^2 + 2.8^2} = \sqrt{73.45}$
$BC = \sqrt{4.3^2 + 2.8^2} = \sqrt{26.33}$
$(AB)^2 = (AC)^2 + (CB)^2 - 2(AC)(BC)\cos(ACB)$
$84.1 = 73.45 + 26.33 - 2\sqrt{73.45}\sqrt{26.33}\cos(ACB)$
Angle $ACB = 80°$

68. $(CS)^2 = R^2 + (ST)^2 - 2(R)(ST)\cos 122.4°$
$(CS)^2 = 3964^2 + 1034^2 - 2(3964)(1034)\cos 122.4°$
$CS \approx 4602$
height $= CS - R \approx 4602 - 3964 \approx 638$ miles

Section 8–3

2. Answers will vary.

4. A vector that has magnitude 1 is called a unit vector.

6. The difference is the vector that represents the wind or current! That is, the apparent velocity is the velocity relative to the air, while the actual velocity is the resultant of the apparent velocity and the wind velocity, that is, the velocity relative to the ground.

8. The coordinates of P are given by
$(x_p, y_p) = (x_b - x_a, y_b - y_a) = (3 - 2, 15 - 7)$
$= (1, 8)$
Hence $\boldsymbol{OP} = \langle 1, 8 \rangle$

10. The coordinates of P are given by
$(x_p, y_p) = (x_b - x_a, y_b - y_a) = (8 - (-5), (-1) - 2)$
$= (13, -3)$
Hence $\boldsymbol{OP} = \langle 13, -3 \rangle$

12. The coordinates of P are given by
$(x_p, y_p) = (x_b - x_a, y_b - y_a) = (0 - 9, 0 - (-7))$
$= (-9, 7)$
Hence $\boldsymbol{OP} = \langle -9, 7 \rangle$

14. $\boldsymbol{OP} = \boldsymbol{AB} = \langle 7, 1 \rangle$

16. $|\boldsymbol{v}| = \sqrt{0^2 + 8^2} = \sqrt{64} = 8$

18. $|\boldsymbol{v}| = \sqrt{(-4)^2 + 3^2} = \sqrt{25} = 5$

20. $|\boldsymbol{v}| = \sqrt{10^2 + (-10)^2} = \sqrt{200} = 10\sqrt{2}$

22. $|\boldsymbol{v}| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$

24. $|\boldsymbol{u} + \boldsymbol{v}| = \sqrt{|\boldsymbol{u}|^2 + |\boldsymbol{v}|^2 - 2|\boldsymbol{u}||\boldsymbol{v}|\cos(180° - \theta)}$

$= \sqrt{120^2 + 84^2 - 2(120)(84)\cos(180° - 44°)}$

$= 190$ gm

$\dfrac{190}{\sin 136°} = \dfrac{84}{\sin\alpha}$

$\alpha = 18°$

26. $|\boldsymbol{u} + \boldsymbol{v}| = \sqrt{|\boldsymbol{u}|^2 + |\boldsymbol{v}|^2 - 2|\boldsymbol{u}||\boldsymbol{v}|\cos(180° - \theta)}$

$= \sqrt{8.0^2 + 2.0^2 - 2(8.0)(2.0)\cos(180° - 64°)}$

$= 9.1$ knots

$\dfrac{2.0}{\sin\alpha} = \dfrac{9.1}{\sin 116°}$

$\alpha = 11°$

28. $\dfrac{33}{\sin 137°} = \dfrac{|\boldsymbol{v}|}{\sin 17°} \Rightarrow |\boldsymbol{v}| = 14$ kg; $\dfrac{|\boldsymbol{u}|}{\sin 26°} = \dfrac{33}{\sin 137°} \Rightarrow |\boldsymbol{u}| = 21$ kg

30. $\dfrac{437}{\sin 129.5°} = \dfrac{|\boldsymbol{u}|}{\sin 32.7°} = \dfrac{|\boldsymbol{v}|}{\sin 17.8°} \Rightarrow |\boldsymbol{u}| = 306$ mph and $|\boldsymbol{v}| = 173$ mph

32. (A) $\boldsymbol{u} + \boldsymbol{v} = \langle -1, 2\rangle + \langle 3, -2\rangle = \langle -1+3, 2+(-2)\rangle = \langle 2, 0\rangle$ (B) $\boldsymbol{u} - \boldsymbol{v} = \langle -1, 2\rangle - \langle 3, -2\rangle = \langle -1-3, 2-(-2)\rangle = \langle -4, 4\rangle$
(C) $2\boldsymbol{u} - \boldsymbol{v} + 3\boldsymbol{w} = 2\langle -1, 2\rangle - \langle 3, -2\rangle + 3\langle 0, -2\rangle = \langle -2-3+0, 4+2-6\rangle = \langle -5, 0\rangle$

34. (A) $\boldsymbol{u} + \boldsymbol{v} = \langle -3, 2\rangle + \langle -2, 2\rangle = \langle -3+(-2), 2+2\rangle = \langle -5, 4\rangle$ (B) $\boldsymbol{u} - \boldsymbol{v} = \langle -3, 2\rangle - \langle -2, 2\rangle = \langle -3-(-2), 2-2\rangle = \langle -1, 0\rangle$
(C) $2\boldsymbol{u} - \boldsymbol{v} + 3\boldsymbol{w} = 2\langle -3, 2\rangle - \langle -2, 2\rangle + 3\langle -3, 0\rangle = \langle -6+2-9, 4-2+0\rangle = \langle -13, 2\rangle$

36. $\langle 2, -5\rangle = \langle 2, 0\rangle + \langle 0, -5\rangle = 2\langle 1, 0\rangle - 5\langle 0, 1\rangle = 2\boldsymbol{i} - 5\boldsymbol{j}$

38. $\langle 0, -27\rangle = -27\langle 0, 1\rangle = -27\boldsymbol{j}$

40. $\overrightarrow{AB} = \langle 0-(-2), 2-(-1)\rangle = \langle 2, 3\rangle = \langle 2, 0\rangle + \langle 0, 3\rangle = 2\langle 1, 0\rangle + 3\langle 0, 1\rangle = 2\boldsymbol{i} + 3\boldsymbol{j}$

42. $\boldsymbol{u} - \boldsymbol{v} = 3\boldsymbol{i} - 2\boldsymbol{j} - (2\boldsymbol{i} + 4\boldsymbol{j}) = 3\boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{i} - 4\boldsymbol{j} = \boldsymbol{i} - 6\boldsymbol{j}$

44. $3\boldsymbol{u} + 2\boldsymbol{v} = 3(3\boldsymbol{i} - 2\boldsymbol{j}) + 2(2\boldsymbol{i} + 4\boldsymbol{j}) = 9\boldsymbol{i} - 6\boldsymbol{j} + 4\boldsymbol{i} + 8\boldsymbol{j} = 13\boldsymbol{i} + 2\boldsymbol{j}$

46. $\boldsymbol{u} - 3\boldsymbol{v} + 2\boldsymbol{w} = 3\boldsymbol{i} - 2\boldsymbol{j} - 3(2\boldsymbol{i} + 4\boldsymbol{j}) + 2(2\boldsymbol{i}) = 3\boldsymbol{i} - 2\boldsymbol{j} - 6\boldsymbol{i} - 12\boldsymbol{j} + 4\boldsymbol{i} = \boldsymbol{i} - 14\boldsymbol{j}$

48. $|\boldsymbol{v}| = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$

$\boldsymbol{u} = \dfrac{1}{|\boldsymbol{v}|}\boldsymbol{v} = \dfrac{1}{13}\langle 5, 12\rangle = \left\langle \dfrac{5}{13}, \dfrac{12}{13}\right\rangle$

50. $|\boldsymbol{v}| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$

$\boldsymbol{u} = \dfrac{1}{|\boldsymbol{v}|}\boldsymbol{v} = \dfrac{1}{\sqrt{13}}\langle 2, -3\rangle = \left\langle \dfrac{2}{\sqrt{13}}, -\dfrac{3}{\sqrt{13}}\right\rangle$

52. $|\boldsymbol{v}| = \sqrt{0^2 + (-17)^2} = \sqrt{289} = 17$

$\boldsymbol{u} = \dfrac{1}{|\boldsymbol{v}|}\boldsymbol{v} = \dfrac{1}{17}\langle 0, -17\rangle = \langle 0, -1\rangle$

54. $|\boldsymbol{v}| = \sqrt{(-\sqrt{2})^2 + (-\sqrt{7})^2} = \sqrt{9} = 3$

$\boldsymbol{u} = \dfrac{1}{|\boldsymbol{v}|}\boldsymbol{v} = \dfrac{1}{3}(-\sqrt{2}\,\boldsymbol{i} - \sqrt{7}\,\boldsymbol{j})$

$= -\dfrac{\sqrt{2}}{3}\boldsymbol{i} - \dfrac{\sqrt{7}}{3}\boldsymbol{j} = \left\langle -\dfrac{\sqrt{2}}{3}, -\dfrac{\sqrt{7}}{3}\right\rangle$

56. True. The standard vector is an exact representation of the vector.
58. False. $-\boldsymbol{i}$ is another unit vector, and there are an infinite number.

60. True. $\boldsymbol{v}+\boldsymbol{v}=2\boldsymbol{v}$ has the same direction as $\boldsymbol{v}$ and twice the magnitude.

62. False. $\boldsymbol{i}+\boldsymbol{j}$ is not a unit vector.

64. $\boldsymbol{u}+\boldsymbol{v}=\langle a,b\rangle+\langle c,d\rangle=\langle a+c,b+d\rangle$
$=\langle c+a,d+b\rangle=\langle c,d\rangle+\langle a,b\rangle=\boldsymbol{v}+\boldsymbol{u}$

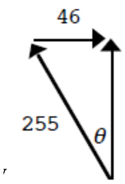
66. $\boldsymbol{u}+(-\boldsymbol{u})=\langle a,b\rangle+(-\langle a,b\rangle)=\langle a,b\rangle+\langle -a,-b\rangle$
$=\langle a+(-a),b+(-b)\rangle=\langle 0,0\rangle=\mathbf{0}$

68. $m(\boldsymbol{u}+\boldsymbol{v})=m(\langle a,b\rangle+\langle c,d\rangle)=m\langle a+c,b+d\rangle$
$=\langle ma+mc,mb+md\rangle=\langle ma,mb\rangle+\langle mc,md\rangle$
$=m\langle a,b\rangle+m\langle c,d\rangle=m\boldsymbol{u}+m\boldsymbol{v}$

70. $1\boldsymbol{u}=1\langle a,b\rangle=\langle 1\cdot a,1\cdot b\rangle=\langle a,b\rangle=\boldsymbol{u}$

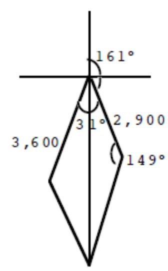
72. $|\bar{v}|=\sqrt{15^2+3.9^2-2(15)(3.9)\cos(25°+45°)}$
≈ 14 mph
$\frac{3.9}{\sin\theta}=\frac{14}{\sin(70°)}\Rightarrow\theta=15°$
heading = 25° + 15° = 40°
14 mph at 40°

74.



$\sqrt{255^2-46^2}=251$
$\sin\theta=\frac{46}{255}$
$\theta=10.4°$
251 mph at 349.6° (360° − 10.4°)

76. $R=\sqrt{3600^2+2900^2-2(3600)(2900)\cos 149°}$
$=6300$
$\frac{6300}{\sin 149°}=\frac{2900}{\sin\alpha}\Rightarrow\alpha=13.71°$
The third angle: 180° − (149° + 14°) = 17°
direction = 161° + 17° = 178°
6300 kg @ 178°



78. (A) parallel force = 2500 sin 15°
= 650 lb
(B) force perpendicular = 2500 cos 15°
= 2400 lb

80. Left: 41 sin 31° = 21;
Right: 31 sin 41° = 20
21 > 20 ⇒ slide left

82. Let the left tension be represented by T_L and the right tension by T_R. Then
$T_L\sin 4.2°+T_R\sin 5.3°=112$
$T_L\cos 4.2°=T_R\cos 5.3°$
Solving the second equation for T_L:
$T_L=T_R\frac{\cos 5.3°}{\cos 4.2°}$
Substituting: $\frac{T_R(\cos 5.3°)(\sin 4.2°)}{\cos 4.2°}+T_R\sin 5.3°=112$
Graph $y_1=\frac{x(\cos 5.3°)(\sin 4.2°)}{\cos 4.2°}+x\sin 5.3°$ and $y_2=112$.
$T_R\approx 677$ lb, $T_L\approx 676$ lb.

84. Let the left tension be represented by T_L and the right tension by T_R. Then
$T_L\sin 45°+T_R\sin 20°=500$
$T_L\cos 45°=T_R\cos 20°$
Solving the second equation for T_L : $T_L=\frac{T_R\cos 20°}{\cos 45°}$
Substituting: $\frac{T_R(\cos 20°)(\sin 45°)}{\cos 45°}+T_R\sin 20°=500$
Graph $y_1=\frac{x(\cos 20°)(\sin 45°)}{\cos 45°}+x\sin 20°$ and $y_2=500$.
$T_R\approx 390$ lb, $T_L\approx 518$ lb

86. Angle $ABC = 30°$
$BC \sin 30° = 1000 \Rightarrow BC = 2000$ kg, tension
$BC \cos 30° = AB \Rightarrow AB = 2000 \cos 30° \approx 1730$ kg, compression

88. Angle ABC: $\cos(ABC) = \frac{5}{6} \Rightarrow ABC \approx 33.6°$ (Answers may vary due to rounding.)
$AB \sin 33.6° = 5000$
$AB \approx 9040$ kg, compression
$BC = AB \cos 33.6° \approx 9040 \cos 33.6° \approx 7530$ kg, tension

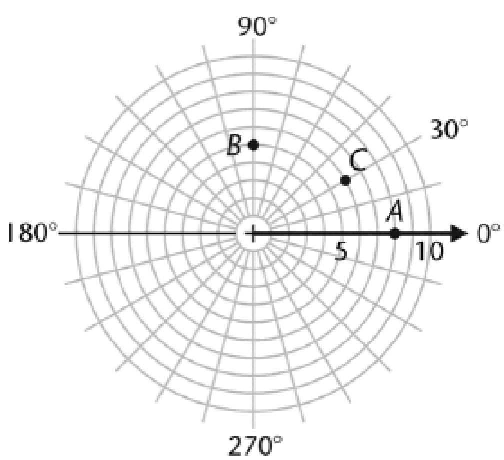
Section 8−4

2. If r is positive, θ represents the angle between the polar axis and the ray connecting the point to the pole; θ may be any of the coterminal such angles. If r is negative θ represents the angle between the polar axis and the ray opposite to the ray connecting the point to the pole; again, θ may be any of the coterminal such angles.

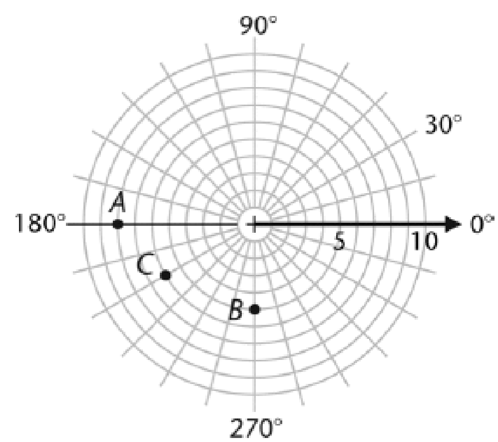
4. One set of polar coordinates is given by $r = \sqrt{x^2 + y^2}$, $\tan \theta = \frac{y}{x}$. If $x = 0$, then θ can be given as $\frac{\pi}{2}$ if y is positive and by $-\frac{\pi}{2}$ or $\frac{3\pi}{2}$ if y is negative.

6. Answers will vary.

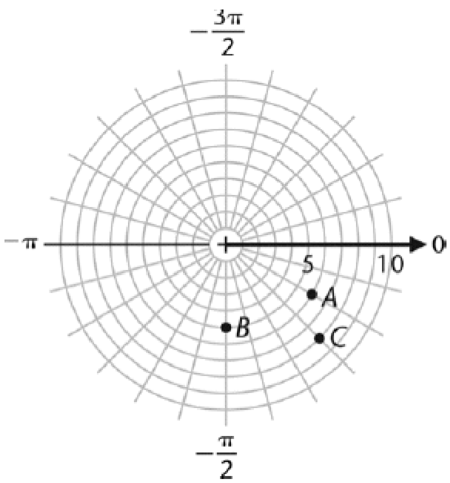
8.



10.



12.



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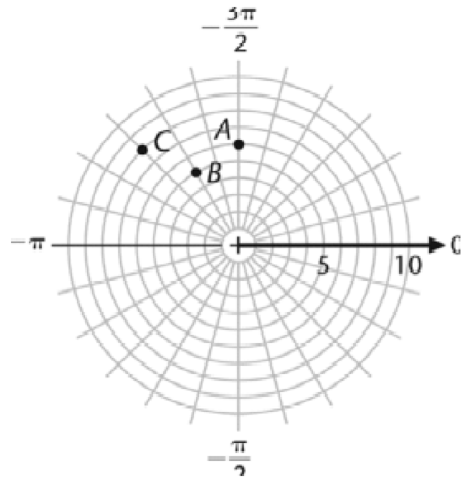
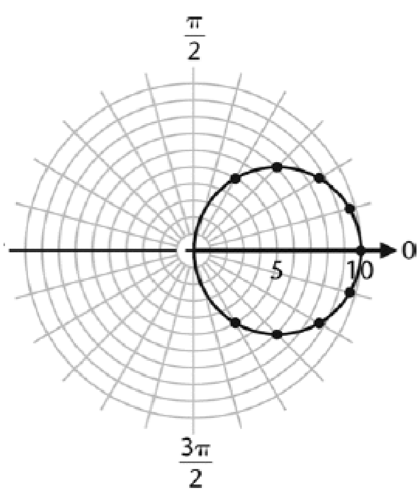


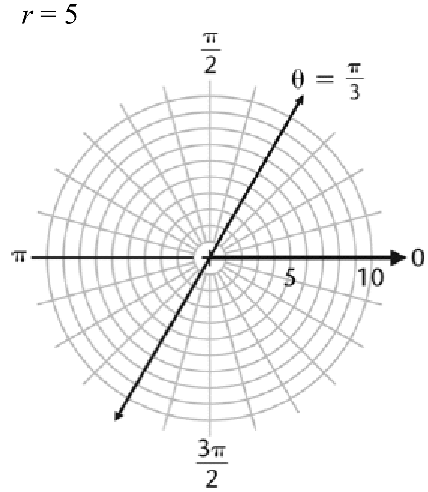
16. $(-6, -210°)$: The polar axis is rotated 210° clockwise (negative direction) and the point is located 6 units from the pole along the negative polar axis.
$(-6, 150°)$: The polar axis is rotated 150° counterclockwise (positive direction) and the point is located 6 units from the pole along the negative polar axis.
$(6, 330°)$: The polar axis is rotated 330° counterclockwise (positive direction) and the point is located 6 units along the positive polar axis.

18.

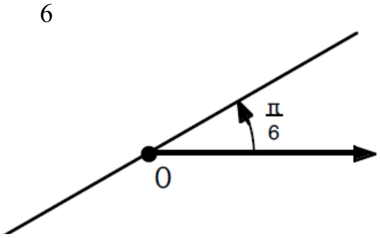
θ	$10\cos\theta$
0	10
$\pi/6$	$5\sqrt{3}$
$\pi/4$	$5\sqrt{2}$
$\pi/3$	5
$\pi/2$	0
$2\pi/3$	-5
$3\pi/4$	$-5\sqrt{2}$
$5\pi/6$	$-5\sqrt{3}$
π	-10



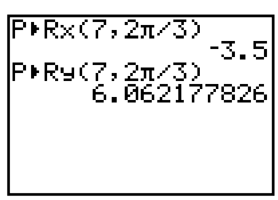
20. $r = 5$



22. $\theta = \frac{\pi}{6}$

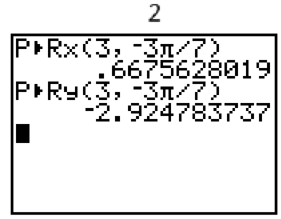


24.

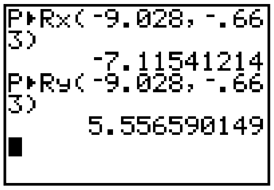


$(-3.500, 6.062)$

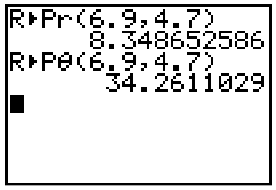
26.



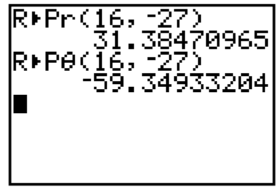
$(0.668, -2.925)$

28. 

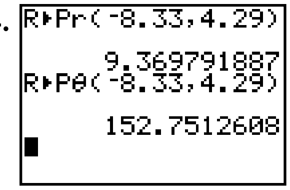
(−7.115, 5.557)

30. 

(8.3, 34°)

32. 

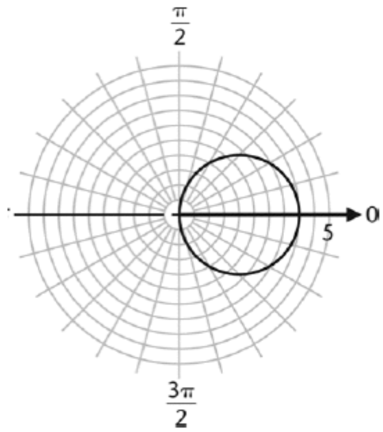
(31, −59°)

34. 

(9.37, 152.8°)

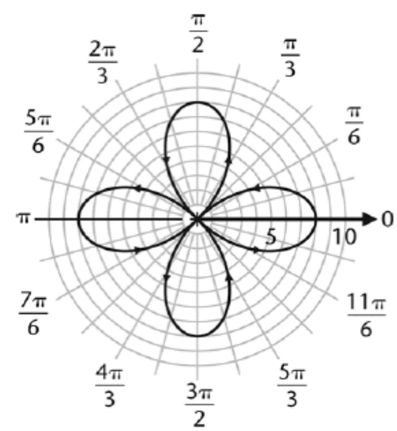
36.

θ varies from	cos θ varies from	4 cos θ varies from
0 to π/2	1 to 0	4 to 0
π/2 to π	0 to −1	0 to −4
π to 3π/2	−1 to 0	−4 to 0
3π/2 to 2π	0 to 1	0 to 4



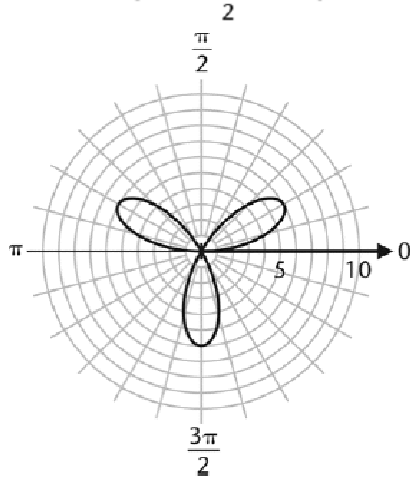
38.

θ varies from	2θ varies from	cos 2θ varies from	8 cos 2θ varies from
0 to π/4	0 to π/2	1 to 0	8 to 0
π/4 to π/2	π/2 to π	0 to −1	0 to −8
π/2 to 3π/4	π to 3π/2	−1 to 0	−8 to 0
3π/4 to π	3π/2 to 2π	0 to 1	0 to 8
π to 5π/4	2π to 5π/2	1 to 0	8 to 0
5π/4 to 3π/2	5π/2 to 3π	0 to −1	0 to −8
3π/2 to 7π/4	3π to 7π/2	−1 to 0	−8 to 0
7π/4 to 2π	7π/2 to 4π	0 to 1	0 to 8



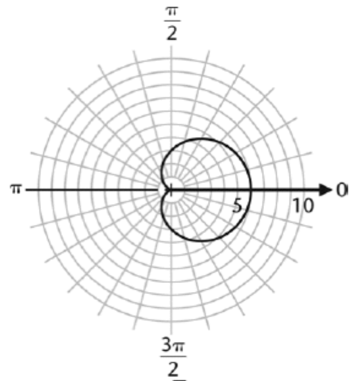
40.

θ varies from	3θ varies from	sin 3θ varies from	6 sin 3θ varies from
0 to π/6	0 to π/2	0 to 1	0 to 6
π/6 to π/3	π/2 to π	1 to 0	6 to 0
π/3 to π/2	π to 3π/2	0 to −1	0 to −6
π/2 to 2π/3	3π/2 to 2π	−1 to 0	−6 to 0
2π/3 to 5π/6	2π to 5π/2	0 to 1	0 to 6
5π/6 to π	5π/2 to 3π	1 to 0	6 to 0
π to 7π/6	3π to 7π/2	0 to −1	0 to −6
7π/6 to 5π/3	7π/2 to 5π	−1 to 0	−6 to 0
M	M	M	M



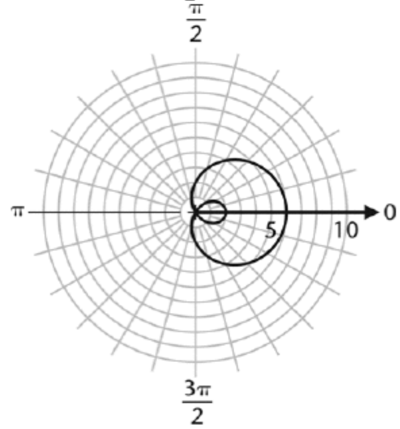
42.

θ varies from	$\cos\theta$ varies from	$3\cos\theta$ varies from	$3+3\cos\theta$ varies from
0 to $\pi/2$	1 to 0	3 to 0	6 to 3
$\pi/2$ to π	0 to –1	0 to –3	3 to 0
π to $3\pi/2$	–1 to 0	–3 to 0	0 to 3
$3\pi/2$ to 2π	0 to 1	0 to 3	3 to 6

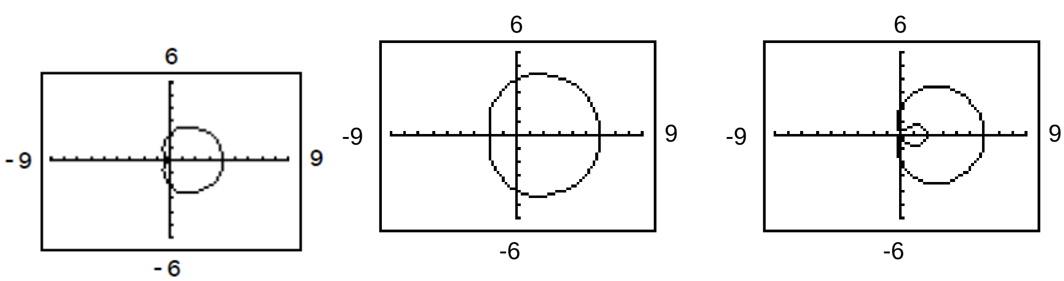


44.

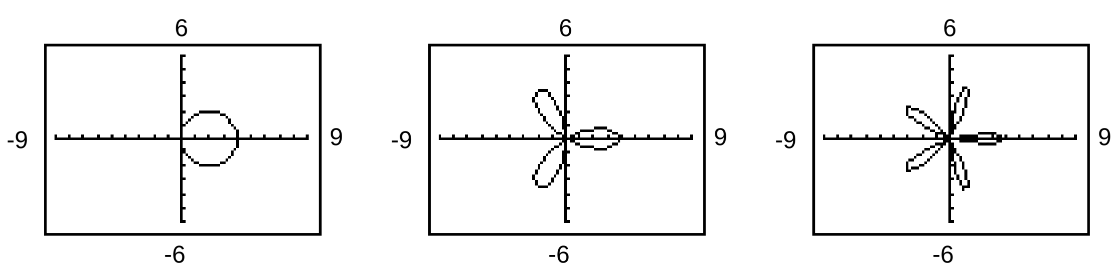
θ varies from	$\cos\theta$ varies from	$4\cos\theta$ varies from	$2+4\cos\theta$ varies from
0 to $\pi/2$	1 to 0	4 to 0	6 to 2
$\pi/2$ to π	0 to –1	0 to –4	2 to –2
π to $3\pi/2$	–1 to 0	–4 to 0	–2 to 2
$3\pi/2$ to 2π	0 to 1	0 to 4	2 to 6



46. $r = 2 + 2\cos\theta$ $\qquad$ $r = 4 + 2\cos\theta$ $\qquad$ $r = 2 + 4\cos\theta$

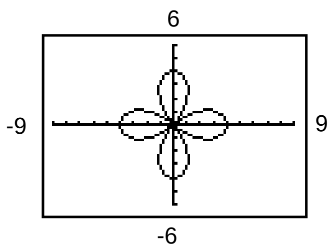


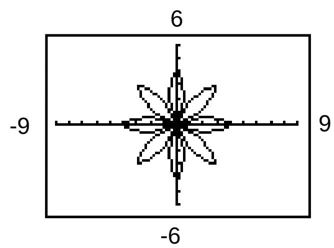
48. (A) $r = 4\cos\theta$ $\qquad$ $r = 4\cos 3\theta$ $\qquad$ $r = 4\cos 5\theta$

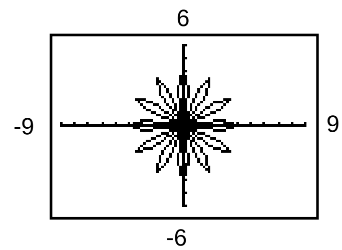


(B) 7 leaves in $r = 4\cos 7\theta$
(C) n leaves in $r = a\cos(n\theta)$ ($a > 0$ and n odd)

50. (A) $r = 4 \cos 2\theta$ $\quad r = 4 \cos 4\theta$ $\quad r = 4 \cos 6\theta$







(B) 16 leaves in $r = 4 \cos 8\theta$

(C) $2n$ leaves in $r = a \cos n\theta$ ($a > 0$ and n even)

52.

$$\begin{aligned} 6x - x^2 &= y^2 \\ 6x &= x^2 + y^2 \\ 6r\cos\theta &= r^2 \\ r &= 6\cos\theta \end{aligned}$$

54.

$$\begin{aligned} x^2 + y^2 &= 9 \\ r^2 &= 9 \\ \text{or } r &= \pm 3 \end{aligned}$$

56.

$$\begin{aligned} 2xy &= 1 \\ 2(r\cos\theta \; r\sin\theta) &= 1 \\ r^2(2\sin\theta\cos\theta) &= 1 \\ r^2 \sin 2\theta &= 1 \\ r^2 &= \frac{1}{\sin 2\theta} \\ r^2 &= \csc 2\theta \end{aligned}$$

58.

$$\begin{aligned} r(2\cos\theta + \sin\theta) &= 4 \\ 2(r\cos\theta) + r\sin\theta &= 4 \\ 2x + y &= 4 \end{aligned}$$

60.

$$\begin{aligned} r &= 8\cos\theta \\ r^2 &= 8r\cos\theta \\ x^2 + y^2 &= 8x \end{aligned}$$

62.

$$\begin{aligned} r &= 4 \\ r^2 &= 16 \\ x^2 + y^2 &= 16 \end{aligned}$$

64.

n	$r = 1 + 2\cos(n\theta)$
1	1 small petal inside 1 large petal
2	2 small petals between 2 large petals
3	3 small petals inside 3 large petals
4	4 small petals between 4 large petals

$r = 1 + 2\cos(n\theta)$ will have n large and n small petals. For n odd the small petals are within the large petals. For n even the small petals are between the large petals.

66. $r = 2\cos\theta$ (1)

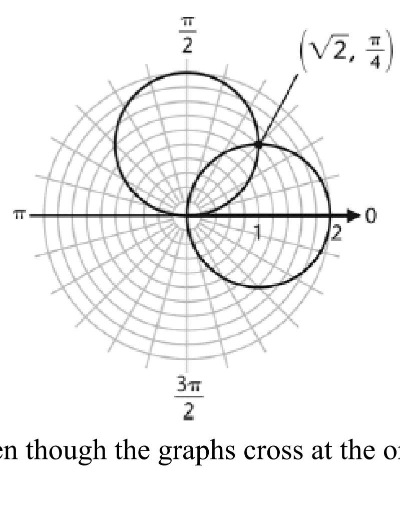
$r = 2\sin\theta$ (2)

$0 \le \theta \le \pi$

Divide (2) by (1):

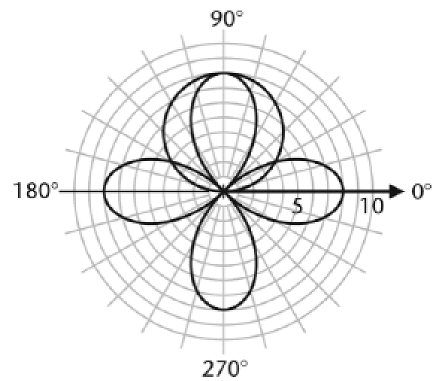
$$1 = \frac{2\sin\theta}{2\cos\theta} = \tan\theta$$

$$\theta = \frac{\pi}{4} \Rightarrow r = \sqrt{2} \qquad \left(\sqrt{2}, \frac{\pi}{4}\right)$$



[Note: (0, 0) is not a solution to this system even though the graphs cross at the origin.]

68. (1) $r = \sin\theta$
(2) $r = \cos 2\theta$
$0° \le \theta \le 360°$
$\sin\theta = \cos 2\theta = 1 - 2\sin^2\theta$
$2\sin^2\theta + \sin\theta - 1 = 0$
$(2\sin\theta - 1)(\sin\theta + 1) = 0$
$2\sin\theta - 1 = 0 \qquad \sin\theta + 1 = 0$
$\sin\theta = \frac{1}{2} \qquad \sin\theta = -1$
$\theta = 30°, 150° \qquad \theta = 270°$
$r = 4, 4 \qquad r = -8$
(4, 30°), (4, 150°), (−8, 270°)
[Note: (0, 0) is not a solution to this system even though the graphs cross at the origin.]

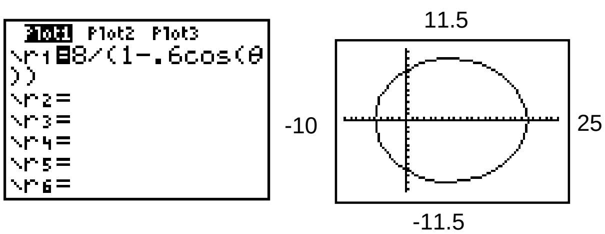


70. $P_1(2, 30°)$ and $P_2(3, 60°)$

$d = \sqrt{(r_1)^2 + (r_2)^2 - 2r_1r_2\cos(\theta_2 - \theta_1)} = \sqrt{2^2 + 3^2 - 2(2)(3)\cos(60° - 30°)} = \sqrt{13 - 12\cos 30°} \approx 1.615$

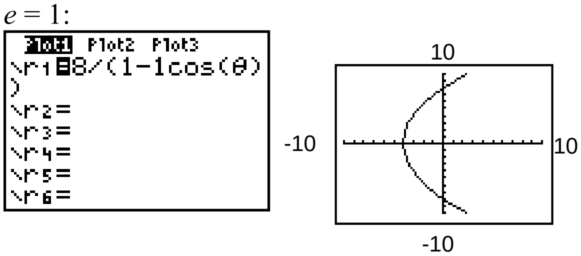
72. at 45°: $9k$, at 90°: $14k$, at 120°: $13k$, at 150°: $11k$

74. (A) $e = 0.6$:



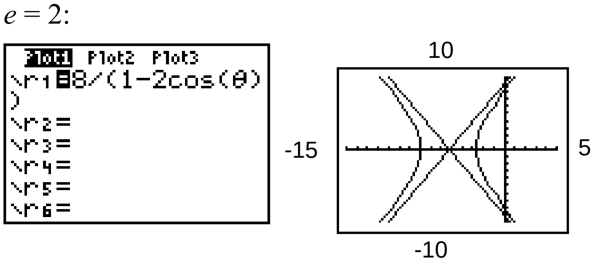
ellipse

(B) $e = 1$:



parabola

(C) $e = 2$:



hyperbola

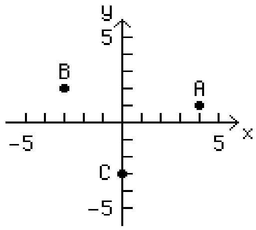
Section 8−5

2. If $z = re^{i\theta}$ then the angle θ is called the argument of z, usually chosen so that $-\pi < \theta \le \pi$ or $-180° < \theta \le 180°$.

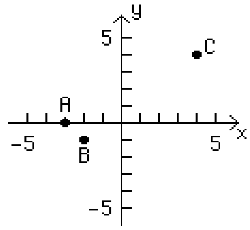
4. If $z_1 = re^{i\theta_1}$ and $z_2 = re^{i\theta_2}$ are the two numbers, then their quotient is $\frac{z_1}{z_2} = e^{i(\theta_1 - \theta_2)}$ which lies on the unit circle with argument $\theta_1 - \theta_2$.

6. If $z = e^{i\theta}$ is the number, then $w_1 = e^{i\theta/3}$, $w_2 = e^{i(\theta+2\pi)/3}$, and $w_3 = e^{i(\theta+4\pi)/3}$ are the cube roots. They lie on the unit circle with arguments $\frac{\theta}{3}$, $\frac{(\theta+2\pi)}{3}$, and $\frac{(\theta+4\pi)}{3}$, that is, 120° apart.

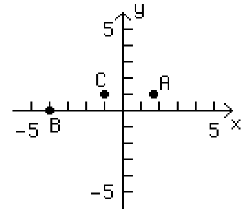
8.



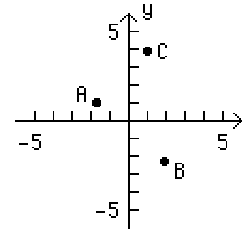
10.



12.



14.



16. (A) $-1 + i\sqrt{3}$

A sketch shows that $-1 + i\sqrt{3}$ is associated with $\theta = 120°$, $r = 2$

$-1 + i\sqrt{3} = 2(\cos 120° + i \sin 120°) = 2e^{120° i}$

(B) $-3i$

A sketch shows that $\theta = -90°$, $r = 3$

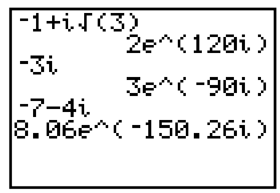
$-3i = 3(\cos(-90°) + i \sin(-90°)) = 3e^{(-90°)i}$

(C) $-7 - 4i$

$\gamma = \sqrt{(-7)^2 + (-4)^2} = \sqrt{65} \approx 8.06$

$\theta = -180° + \tan^{-1}\left(\frac{4}{7}\right) \approx -150.26°$

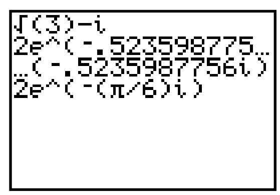
$-7 - 4i \approx 8.06(\cos(-150.26°) + i \sin(-150.26°)) \approx 8.06e^{(-150.26°)i}$



18. (A) $\sqrt{3} - i$

A sketch shows that $\theta = -\frac{\pi}{6}$, $r = 2$

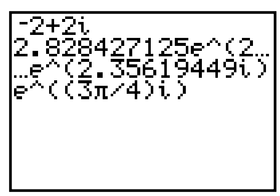
$\sqrt{3} - i = 2\left(\cos\left(-\frac{\pi}{6}\right)\right) + i \sin\left(-\frac{\pi}{6}\right) = 2e^{-(\pi/6)i}$



(B) $-2 + 2i$

A sketch shows that $\theta = \frac{3\pi}{4}$, $r = \sqrt{8} = 2\sqrt{2}$

$-2 + 2i = 2\sqrt{2}\left(\cos\frac{3\pi}{4} + i \sin\frac{3\pi}{4}\right) = 2\sqrt{2}\, e^{(3\pi/4)i}$

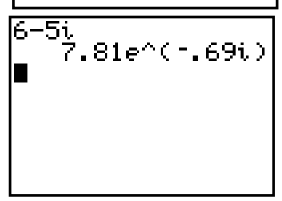


(C) $6 - 5i$

$r = \sqrt{6^2 + (-5)^2} = \sqrt{61} \approx 7.81$

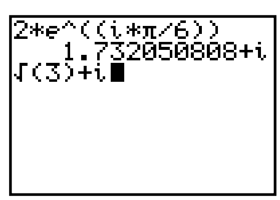
$\theta = \tan^{-1}\left(-\frac{5}{6}\right) \approx -0.69$

$6 - 5i \approx 7.81(\cos(-0.69) + i \sin(-0.69)) \approx 7.81e^{-0.69i}$



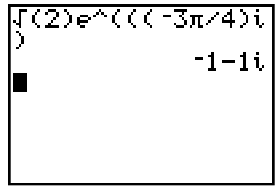
20. (A) $2e^{30° i} = 2(\cos 30° + i \sin 30°) = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \sqrt{3} + i$

$$30° = \frac{\pi}{6}$$

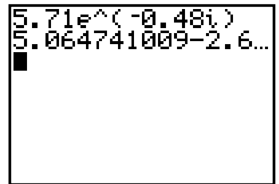


(B) $\sqrt{2}\, e^{(-3\pi/4)i} = \sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$

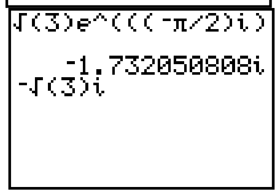
$= \sqrt{2}\left(-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right) = -1 - i$



(C) $5.71e^{(-0.48)i} = 5.71(\cos(-0.48) + i\sin(-0.48)) \approx 5.06 - 2.64i$



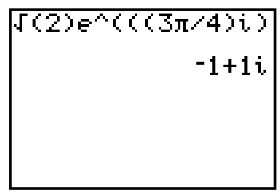
22. (A) $\sqrt{3}\, e^{(-\pi/2)i} = \sqrt{3}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right) = \sqrt{3}\,(0 - i) = -i\sqrt{3}$



(B) $\sqrt{2}\, e^{135°i} = \sqrt{2}\,(\cos 135° + i\sin 135°) = \sqrt{2}\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$

$= -1 + i$

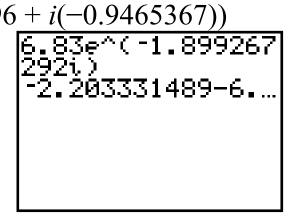
$135° = \frac{3\pi}{4}$



(C) $6.83e^{(-108.82°)i} = 6.83(\cos(-108.82°) + i\sin(-108.82°)) \approx 6.83(-0.322596 + i(-0.9465367))$

$\approx -2.20 - 6.46i$

$-108.82° = -1.899267292$



24. $z_1z_2 = r_1e^{i\theta_1} \cdot r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$

$z_1z_2 = 6e^{132°i} \cdot 3e^{93°i} = 18e^{225°i}$

$\frac{z_1}{z_2} = \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$

$\frac{z_1}{z_2} = \frac{6e^{132°i}}{3e^{93°i}} = 2e^{39°i}$

26. $z_1z_2 = r_1e^{i\theta_1} \cdot r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$

$z_1z_2 = 3e^{67°i} \cdot 2e^{97°i} = 6e^{164°i}$

$\frac{z_1}{z_2} = \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$

$\frac{z_1}{z_2} = \frac{3e^{67°i}}{2e^{97°i}} = 1.5e^{(-30°)i}$

28. $z_1z_2 = r_1e^{i\theta_1} \cdot r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$

$z_1z_2 = 7.11e^{0.79i} \cdot 2.66e^{1.07i} = 18.9126e^{1.86i} \approx 18.91e^{1.86i}$

$\frac{z_1}{z_2} = \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$

$\frac{z_1}{z_2} = \frac{7.11e^{0.79i}}{2.66e^{1.07i}} \approx 2.67e^{(-0.28)i}$

30. $z^n = r^ne^{n\theta i}$

$(5e^{15°i})^3 = 5^3e^{(3\cdot 15°)i} = 125e^{45°i}$

32. $z^n = r^ne^{n\theta i}$

$(\sqrt{2}\, e^{15°i})^8 = (\sqrt{2}\,)^8e^{(8\cdot 15°)i} = 16e^{120°i}$

34. $(\sqrt{3}+i)$: $r = 2$, $\theta = 30° \Rightarrow 2e^{30°i}$

$z^n = r^n e^{n\theta i}$

$(\sqrt{3}+i)^8 = (2e^{30°i})^8 = 2^8 e^{(8\cdot 30°)i} = 256e^{240°i}$

36. $(-1+i)$: $r = \sqrt{2}$, $\theta = 135° \Rightarrow \sqrt{2}\, e^{135°i}$

$z^n = r^n e^{n\theta i}$

$(\sqrt{2}\, e^{135°i})^4 = (\sqrt{2}\,)^4\, e^{(4\cdot 135°)i} = 4e^{540°i}$

$= 4(\cos 180° + i \sin 180°)$

$= 4(-1 + 0i) = -4$

38. $(-\sqrt{3}+i)$: $r = 2$, $= \theta\ 150° \Rightarrow 2e^{150°i}$

$z^n = r^n e^{n\theta i}$

$(2e^{150°i})^5 = 2^5 e^{(5\cdot 150°)i} = 32e^{750°i} = 32e^{30°i}$

$= 32(\cos 30° + i \sin 30°)$

$= 32\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = 16\sqrt{3} + 16i$

40. $\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right)$: $r = 1$, $\theta = 240° \Rightarrow 1e^{240°i}$

$z^n = r^n e^{n\theta i}$

$(1e^{240°i})^3 = 1^3 e^{(3\cdot 240°)i} = 1e^{720°i} = 1e^{0°i}$

$= \cos 0° + i \sin 0° = 1$

42. $z^{1/n} = r^{1/n} e^{[(\theta/n)\ +\ ((360°k)/n)\ ]i}$

$(8e^{45°i})^{1/3} = 8^{1/3} e^{[(45°/3) + ((360°k)/3)]i} = 2e^{(15° + 120°k)i}$

$w_1 = 2e^{15°i}$

$w_2 = 2e^{(15° + 120°)i} = 2e^{135°i}$

$w_3 = 2e^{(15° + 240°)i} = 2e^{255°i}$

44. $z^{1/n} = r^{1/n} e^{[(\theta/n) + ((360°k)/n)]i}$

$(16e^{90°i})^{1/4} = 16^{1/4} e^{[(90°/4) + ((360°k)/4)]i} = 2e^{(22.5° + 90°k)i}$

$w_1 = 2e^{22.5°i}$

$w_2 = 2e^{112.5°i}$

$w_3 = 2e^{202.5°i}$

$w_4 = 2e^{292.5°i}$

46. $(-1+i)$: $r = \sqrt{2}$, $\theta = 135° \Rightarrow \sqrt{2}\, e^{135°i}$

$z^{1/n} = r^{1/n} e^{[(\theta/n)\ +\ ((360°k)/n)\ ]i}$

$(\sqrt{2}\, e^{135°i})^{1/3} = (2^{1/2})^{1/3} e^{[(135°/3) + ((360°k)/3)]i} = 2^{1/6}\, e^{(45° + 120°k)i}$

$w_1 = 2^{1/6}\, e^{45°i}$

$w_2 = 2^{1/6}\, e^{165°i}$

$w_3 = 2^{1/6}\, e^{285°i}$

48. $z = 1 = 1e^{0°i}$

$z^{1/n} = r^{1/n} e^{[(\theta/n) + ((360°k/n)]i}$

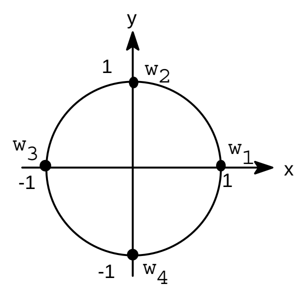
$(1e^{0°i})^{1/4} = 1^{1/4}\, e^{[0° + ((360°k)/4)]i} = 1e^{90°ki}$

$w_1 = 1e^{0°i} = 1 + 0i$

$w_2 = 1e^{90°i} = 0 + i$

$w_3 = 1e^{180°i} = -1 + 0i$

$w_4 = 1e^{270°i} = 0 - i$



50. $z = -8 = 8e^{180°i}$

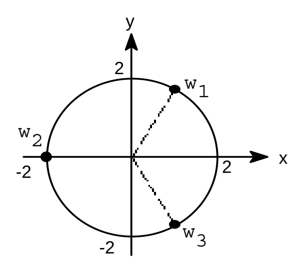
$z^{1/n} = r^{1/n} e^{[(\theta/n) + ((360°k)/n]i}$

$(8e^{180°i})^{1/3} = 8^{1/3}\, e^{[(180°/3) + ((360°k)/3)]i} = 2e^{(60° + 120°k)i}$

$w_1 = 2e^{60°i} = 2(\cos 60° + i \sin 60°) = 1 + \sqrt{3}\, i$

$w_2 = 2e^{180°i} = 2(\cos 180° + i \sin 180°) = -2$

$w_3 = 2e^{300°i} = 2(\cos 300° + i \sin 300°) = 1 - \sqrt{3}\, i$



52. $z = -i = e^{-90°i}$

$z^{1/n} = r^{1/n} e^{[(\theta/n) + ((360°k)/n)]i}$

$(1e^{-90°})^{1/5} = 1^{1/5}\, e^{[(-90°/5) + ((360°k)/5]i} = 1e^{(-18° + 72°k)i}$

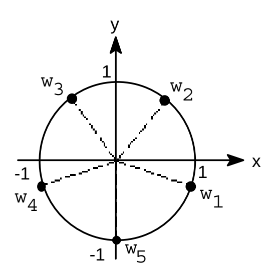
$w_1 = 1e^{(-18°)i}$

$w_2 = 1e^{54°i}$

$w_3 = 1e^{126°i}$

$w_4 = 1e^{198°i}$

$w_5 = 1e^{270°i}$



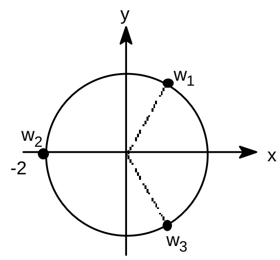
54. (A) $x^3 + 8 = 0$

$(-2)^3 + 8 = 0$

$-8 + 8 = 0$, -2 is a root of $x^3 + 8 = 0$

$x^3 + 8 = 0$ is degree 3 so there are two more roots.

(B)



$\frac{360°}{3} = 120°$ is the spacing between roots

From problem 50, $w_1 = 1 + \sqrt{3}\, i$, $w_2 = 1 - \sqrt{3}\, i$

(C) $(1 + \sqrt{3}\, i)^3 + 8 = [1 + 3(\sqrt{3}\, i) + 3(\sqrt{3}\, i)^2 + (\sqrt{3}\, i)^3] + 8 = [1 + 3\sqrt{3}\, i - 9 - 3\sqrt{3}\, i] + 8$
$= -8 + 8 = 0$

(D) In the same manner, $(1 - \sqrt{3}\, i)^3 + 8 = -8 + 8 = 0$

56. $x^3 - 64 = 0$

$x^3 = 64$

$x = 64^{1/3} = (64e^{0°i})^{1/3}$

$z^{1/n} = r^{1/n} e^{[(\theta/n) + ((360°k)/n)]i}$

$(64e^{0°i})^{1/3} = 64^{1/3}\, e^{[(0°/3) + ((360°k)/3)]i} = 4e^{120°ki}$

$x_1 = 4e^{0°i} = 4$

$x_2 = 4e^{120°i} = 4(\cos 120° + i \sin 120°) = 4\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) = -2 + 2\sqrt{3}\, i$

$x_3 = 4e^{240°i} = 4(\cos 240° + i \sin 240°) = 4\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = -2 - 2\sqrt{3}\, i$

58. $x^3 + 27 = 0$

$x^3 = -27$

$x = (-27)^{1/3} = 27e^{180°i}$

$z^{1/n} = r^{1/n} e^{[(\theta/n) + ((360°k)/n)]i}$

$(27e^{180°i})^{1/3} = 27^{1/3}\, e^{[(180°/3) + ((360°k)/n)]i} = 3e^{(60° + 120°k)i}$

$x_1 = 3e^{60°i} = 3(\cos 60° + i \sin 60°) = 3\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) = \frac{3}{2} + \frac{3\sqrt{3}}{2} i$

$x_2 = 3e^{180°i} = 3(\cos 180° + i \sin 180°) = 3(-1 + 0i) = -3$

$x_3 = 3e^{300°i} = 3(\cos 300° + i \sin 300°) = 3\left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = \frac{3}{2} - \frac{3\sqrt{3}}{2} i$

60. False. If $z_1 = r_1 e^{i90°}$ and $z_2 = r_2 e^{i90°}$, then $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i0°}$.

62. True. The square roots of $re^{i0°}$ are $\sqrt{r}\, e^{i0°}$ and $\sqrt{r}\, e^{i180°}$, which are the real numbers $\sqrt{r}$ and $-\sqrt{r}$.

64. False. $e^{i60°}$ is a sixth root of 1, but $(e^{i60°})^2 = e^{i120°}$ is not 1.

66. $\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2} \cdot \frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2} \cdot \frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}$

$= \frac{r_1}{r_2} \cdot \frac{\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\cos\theta_2\sin\theta_1 - i^2\sin\theta_1\sin\theta_2}{\cos^2\theta_2 - i^2\sin^2\theta_2}$

$= \frac{r_1}{r_2} \cdot \frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\cos\theta_2\sin\theta_1 - \cos\theta_1\sin\theta_2)}{1} = \frac{r_1}{r_2} \cdot [\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

68. For $k = 0$, $r^{1/n}e^{(\theta/n + (k\cdot 360°)/n)i} = r^{1/n}e^{(\theta/n)i}$
For $k = n$, $r^{1/n}e^{(\theta/n + (k\cdot 360°)/n)i} = r^{1/n}e^{(\theta/n + 360°)i} = r^{1/n}e^{(\theta/n)i}$

70.
$$x^6 + 1 = 0$$
$$x^6 = -1$$
$$x = (-1)^{1/6} = 1e^{180°i}$$
$$z^{1/n} = r^{1/n}e^{[(\theta/n) + ((360°k)/n)]i}$$
$$(1e^{180°i})^{1/6} = 1^{1/6}\,e^{[(180°/6) + ((360°k)/6)]i} = 1e^{(30° + 60°k)i}$$
$x_1 = 1e^{30°i}$
$x_2 = 1e^{90°i}$
$x_3 = 1e^{150°i}$
$x_4 = 1e^{210°i}$
$x_5 = 1e^{270°i}$
$x_6 = 1e^{330°i}$

72.
$$x^3 - i = 0$$
$$x^3 = i$$
$$x = i^{1/3} = 1e^{90°i}$$
$$z^{1/n} = r^{1/n}e^{[(\theta/n) + ((360°k)/n)]i}$$
$$(1e^{90°i})^{1/3} = 1^{1/3}\,e^{[(90°/3) + ((360°k)/3)]i} = 1e^{(30° + 120°k)i}$$
$x_1 = 1e^{30°i}$
$x_2 = 1e^{150°i}$
$x_3 = 1e^{270°i}$

74. $P(x) = x^6 - 1$; find $x = 1^{1/6} = (1e^{0°i})^{1/6}$ and write as factors.
$$z^{1/n} = r^{1/n}e^{[(\theta/n) + ((360°k)/n)]i}$$
$$(1e^{0°i})^{1/6} = 1^{1/6}\,e^{[(0°/6) + ((360°k)/6)]i} = 1e^{60°ki}$$
$x_1 = e^{0°i} = 1$

$x_2 = e^{60°i} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$x_3 = e^{120°i} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$x_4 = e^{180°i} = -1$

$x_5 = e^{240°i} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

$x_6 = e^{300°i} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

$$P(x) = x^6 + 1 = (x - 1)(x + 1)\left(x - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)\left(x - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)\cdot\left(x - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)\left(x - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)$$

Chapter 8 Group Activity

Problem 1: In the figure, the distance from P to F is r. Since P has rectangular coordinates $(r\cos\theta, r\sin\theta)$ and Q has rectangular coordinates $(-p, r\sin\theta)$, the distance from P to d is $r\cos\theta - (-p) = r\cos\theta + p$.

Then using $PF = e\cdot PD$ as stated:
$$r = e(r\cos\theta + p)$$
$$r = er\cos\theta + ep$$
$$r(1 - e\cos\theta) = ep$$
$$r = \frac{ep}{1 - e\cos\theta}$$

Problems 2−4 are left to the student.
If $0 < e < 1$, the conic section is an ellipse,
If $e = 1$, the conic section is a parabola.
If $e > 1$, the conic section is a hyperbola.