CHAPTER 10

Section 10–1

1. Graph the two equations in the same coordinate system. Determine the point of intersection (if shown) by inspection. Check by substitution in both equations.

3. By multiplying both sides of the equations by non–zero constants as required, match coefficients of one variable so that they are equal in absolute value and opposite in sign. Add to eliminate one variable, solve the resulting equation for the other, and substitute into the original equations to find the other variable, and check.

5. No. A system of linear equations can have no solution, one solution, or infinitely many solutions.

7. Both lines in the given system are different, but they have the same slope $\left(\frac{1}{2}\right)$ and are therefore parallel.

This system corresponds to (b) and has no solution.

9. In slope–intercept form, these equations are $y = 2x - 5$ and $y = -\frac{3}{2}x - \frac{3}{2}$. Thus, one has slope 2 and y intercept –5; the other has slope $-\frac{3}{2}$ and y intercept $-\frac{3}{2}$. This system corresponds to (d) and its solution can be read from the graph as (1, –3). Checking, we see that

$$\begin{aligned} 2x - y &= 2\cdot 1 - (-3) = 5 \\ 3x + 2y &= 3\cdot 1 + 2(-3) = -3 \end{aligned}$$

Note: Checking steps are not shown, but should be performed by the student.

11. $x + y = 7$
$x - y = 3$
If we add, we can eliminate y.

$$\begin{aligned} x + y &= 7 \\ x - y &= 3 \\ \hline 2x &= 10 \\ x &= 5 \end{aligned}$$

Now substitute $x = 5$ back into the top equation and solve for y.

$$\begin{aligned} 5 + y &= 7 \\ y &= 2 \end{aligned}$$

(5, 2)

13. $3x - 2y = 12$
$7x + 2y = 8$
If we add, we can eliminate y.

$$\begin{aligned} 3x - 2y &= 12 \\ 7x + 2y &= 8 \\ \hline 10x &= 20 \\ x &= 2 \end{aligned}$$

Now substitute $x = 2$ back into the bottom equation and solve for y.

$$\begin{aligned} 7(2) + 2y &= 8 \\ 2y &= -6 \\ y &= -3 \end{aligned}$$

(2, –3)

15. $3u + 5v = 15$
$6u + 10v = -30$
If we multiply the top equation by –2 and add, we eliminate both u and v.

$$\begin{aligned} -6u - 10v &= -30 \\ 6u + 10v &= -30 \\ \hline 0 &= -60 \end{aligned}$$

No solution. The equations represent parallel lines.

17. $3x - y = -2$
$-9x + 3y = 6$
If we multiply the top equation by 3 and add, we eliminate both x and y.

$$\begin{aligned} 9x - 3y &= -6 \\ -9x + 3y &= 6 \\ \hline 0 &= 0 \end{aligned}$$

The system is dependent and has infinite solutions. Solving the first equation for y in terms of x, we obtain $y = 3x + 2$. Thus if we let $x = s$, $y = 3s + 2$, we can express the solution set as
$\{(s, 3s + 2) \mid s \text{ any real number}\}$

19. $x - y = 4$
$x + 3y = 12$
Solve the first equation for x in terms of y.
$x = 4 + y$
Substitute into the second equation to eliminate x.
$(4 + y) + 3y = 12$
$4y = 8$
$y = 2$
Now replace y with 2 in the first equation to find x.
$x - 2 = 4$
$x = 6$
Solution: $x = 6, y = 2$

21. $4x + 3y = 26$
$3x - 11y = -7$
Solve the second equation for x in terms of y.
$3x = 11y - 7$
$$x = \frac{11y-7}{3}$$
Substitute into the first equation to eliminate x.
$$4\left(\frac{11y-7}{3}\right) + 3y = 26$$
$$\frac{44y-28}{3} + 3y = 26$$
$$44y - 28 + 9y = 78$$
$$53y = 106$$
$$y = 2$$
$$x = \frac{11\cdot 2-7}{3}$$
$$x = 5$$
Solution: $x = 5, y = 2$

23. $7m + 12n = -1$
$5m - 3n = 7$
Solve the first equation for n in terms of m.
$12n = -1 - 7m$
$$n = \frac{-1-7m}{12}$$
Substitute into the second equation to eliminate n.
$$5m - 3\left(\frac{-1-7m}{12}\right) = 7$$
$$5m - \frac{-1-7m}{4} = 7$$
$$20m + 1 + 7m = 28$$
$$27m = 27$$
$$m = 1$$
$$n = \frac{-1-7(1)}{12}$$
$$n = -\frac{2}{3}$$
Solution: $m = 1, n = -\frac{2}{3}$

25. $y = 0.08x$
$y = 100 + 0.04x$
Substitute y from the first equation into the second equation to eliminate y.
$0.08x = 100 + 0.04x$
$0.04x = 100$
$x = 2{,}500$
$y = 0.08(2{,}500)$
$y = 200$
Solution: $x = 2{,}500, y = 200$

27. $\frac{2}{5}x + \frac{3}{2}y = 2$

$\frac{7}{3}x - \frac{5}{4}y = -5$

Eliminate fractions by multiplying both sides of the first equation by 10 and both sides of the second equation by 12.

$$10\left(\frac{2}{5}x+\frac{3}{2}y\right) = 20$$
$$4x + 15y = 20$$
$$12\left(\frac{7}{3}x-\frac{5}{4}y\right) = -60$$
$$28x - 15y = -60$$

Solve the first equation for y in terms of x and substitute into the second equation to eliminate y.

$$15y = 20 - 4x$$
$$y = \frac{20-4x}{15}$$
$$28x - 15\left(\frac{20-4x}{15}\right) = -60$$
$$28x - (20 - 4x) = -60$$
$$28x - 20 + 4x = -60$$
$$32x = -40$$
$$x = -\frac{5}{4}$$
$$y = \frac{20-4\left(-\frac{5}{4}\right)}{15} = \frac{20+5}{15} = \frac{5}{3}$$

Solution: $x = -\frac{5}{4}$, $y = \frac{5}{3}$

29. $-2.3y + 4.1z = -14.21$

$10.1y - 2.9z = 26.15$

If we multiply the top equation by 2.9 and the bottom equation by 4.1, and add, we can eliminate z.

$$-6.67y + 11.89z = -41.209$$
$$\underline{41.41y - 11.89z = 107.215}$$
$$34.74y = 66.006$$
$$y = 1.9$$

Now substitute $y = 1.9$ back into the first equation and solve for z.

$$-2.3(1.9) + 4.1z = -14.21$$
$$-4.37 + 4.1z = -14.21$$
$$4.1z = -9.84$$
$$z = -2.4$$

31.
$$-2x = 2 \quad E_1$$
$$x - 3y = 2 \quad E_2$$
$$-x + 2y + 3z = -7 \quad E_3$$

Solve E_1 for x.

$$-2x = 2 \quad E_1$$
$$x = -1$$

Substitute $x = -1$ in E_2 and solve for y.

$$x - 3y = 2 \quad E_2$$
$$-1 - 3y = 2$$
$$y = -1$$

Substitute $x = -1$ and $y = -1$ in E_3 and solve for z.

$$-x + 2y + 3z = -7 \quad E_3$$
$$-(-1) + 2(-1) + 3z = -7$$
$$z = -2 \qquad (-1, -1, -2)$$

33.
$$2y - z = 2 \quad E_1$$
$$-4y + 2z = 1 \quad E_2$$
$$x - 2y + 3z = 0 \quad E_3$$

Multiply E_1 by 2 and add to E_2

$$4y - 2z = 4 \quad 2E_1$$
$$\underline{-4y + 2z = 1} \quad E_2$$
$$0 = 5 \quad E_4$$

A contradiction. No solution.

35.
$$x - 3y = 2 \quad E_1$$
$$2y + z = -1 \quad E_2$$
$$x - y + z = 1 \quad E_3$$

Multiply E_1 by -1 and add to E_3 to eliminate x.

$$-x + 3y = -2 \quad (-1)E_1$$
$$\underline{x - y + z = 1} \quad E_3$$
$$2y + z = -1 \quad E_4$$

Equivalent system:

$$x - 3y = 2 \quad E_1$$
$$2y + z = -1 \quad E_2$$
$$2y + z = -1 \quad E_4$$

If E_2 is multiplied by -1 and added to E_4, $0 = 0$ results. The system is dependent and equivalent to

$$x - 3y = 2$$
$$2y + z = -1$$

Let $y = s$. Then

$$2s + z = -1$$
$$z = -2s - 1$$
$$x - 3s = 2$$
$$x = 3s + 2$$

Solutions: $\{(3s + 2, s, -2s - 1) \mid s \text{ is any real number}\}$

37. $2x + z = -5$ E_1
$x - 3z = -6$ E_2
$4x + 2y - z = -9$ E_3

Multiply E_1 by 3 and add to E_2 to eliminate z.

$6x + 3z = -15$ $3E_1$
$x - 3z = -6$ E_2

$7x = -21$ E_4
$x = -3$

Substitute $x = -3$ into E_1 and solve for z.

$2x + z = -5$ E_1
$2(-3) + z = -5$
$z = 1$

Substitute $x = -3$ and $z = 1$ into E_3 and solve for y.

$4(-3) + 2y - 1 = -9$
$y = 2$

$(-3, 2, 1)$

39. $x - y + z = 1$ E_1
$2x + y + z = 6$ E_2
$7x - y + 5z = 15$ E_3

Multiply E_1 by -2 and add to E_2 to eliminate x.
Also multiply E_1 by -7 and add to E_3 to eliminate x.

$-2x + 2y - 2z = -2$ $(-2)E_1$
$\underline{2x + y + z = 6}$ E_2
$3y - z = 4$ E_4
$-7x + 7y - 7z = -7$ $(-7)E_1$
$\underline{7x - y + 5z = 15}$ E_3
$6y - 2z = 8$ E_5

Equivalent system:

$x - y + z = 1$ E_1
$3y - z = 4$ E_4
$6y - 2z = 8$ E_5

If E_4 is multiplied by -2 and added to E_5, $0 = 0$ results. The system is dependent and equivalent to

$x - y + z = 1$
$3y - z = 4$

Let $y = s$. Then $3s - z = 4$
$z = 3s - 4$
$x - s + (3s - 4) = 1$
$x = -2s + 5$

Solutions: $\{(-2s + 5, s, 3s - 4) \mid s \text{ is any real number}\}$

41. $2a + 4b + 3c = -6$ E_1
$a - 3b + 2c = -15$ E_2
$-a + 2b - c = 9$ E_3

Add E_2 to E_3 to eliminate a. Also multiply E_2 by -2 and add to E_1 to eliminate a.

$a - 3b + 2c = -15$ E_2
$\underline{-a + 2b - c = 9}$ E_3
$-b + c = -6$ E_4
$-2a + 6b - 4c = 30$ $(-2)E_2$
$\underline{2a + 4b + 3c = -6}$ E_1
$10b - c = 24$ E_5

Equivalent system:

$a - 3b + 2c = -15$ E_2
$-b + c = -6$ E_4
$10b - c = 24$ E_5

Add E_4 to E_5 to eliminate c

$-b + c = -6$ E_4
$\underline{10b - c = 24}$ E_5
$9b = 18$
$b = 2$

Substitute $b = 2$ into E_4 and solve for c.

$-b + c = -6$ E_4
$-2 + c = -6$
$c = -4$

Substitute $b = 2$ and $c = -4$ into E_2 and solve for a.

$a - 3b + 2c = -15$ E_2
$a - 3(2) + 2(-4) = -15$
$a = -1$

$(-1, 2, -4)$

43. $2x - 3y + 3z = -5$ E_1

Multiply E_4 by -1 and add to E_5 to eliminate z.

$3x + 2y - 5z = 34 \quad E_2$
$5x - 4y - 2z = 23 \quad E_3$

Multiply E_1 by $-\frac{3}{2}$ and add to E_2 to eliminate x.

Also multiply E_1 by $-\frac{5}{2}$ and add to E_3 to eliminate x.

$$-3x + \frac{9}{2}y - \frac{9}{2}z = \frac{15}{2} \quad \left(-\frac{3}{2}\right)E_1$$
$$3x + 2y - 5z = 34 \quad E_2$$
$$\frac{13}{2}y - \frac{19}{2}z = \frac{83}{2} \quad E_4$$
$$-5x + \frac{15}{2}y - \frac{15}{2}z = \frac{25}{2} \quad \left(-\frac{5}{2}\right)E_1$$
$$5x - 4y - 2z = 23 \quad E_3$$
$$\frac{7}{2}y - \frac{19}{2}z = \frac{71}{2} \quad E_5$$

Equivalent system:

$$2x - 3y + 3z = -5 \quad E_1$$
$$\frac{13}{2}y - \frac{19}{2}z = \frac{83}{2} \quad E_4$$
$$\frac{7}{2}y - \frac{19}{2}z = \frac{71}{2} \quad E_5$$

$$-\frac{13}{2}y + \frac{19}{2}z = -\frac{83}{2} \quad (-1)E_4$$
$$\frac{7}{2}y - \frac{19}{2}z = \frac{71}{2} \quad E_5$$
$$-3y = -6 \quad E_6$$

Solve E_6 for y to obtain $y = 2$. Substitute $y = 2$ into E_4 and solve for z.

$$\frac{13}{2}y - \frac{19}{2}z = \frac{83}{2} \quad E_4$$
$$\frac{13}{2}(2) - \frac{19}{2}z = \frac{83}{2}$$
$$-\frac{19}{2}z = \frac{57}{2}$$
$$z = -3$$

Substitute $y = 2$ and $z = -3$ into E_1 and solve for x.

$$2x - 3y + 3z = -5 \quad E_1$$
$$2x - 3(2) + 3(-3) = -5$$
$$x = 5$$

(5, 2, –3)

45. $-x + 2y - z = -4 \quad E_1$
$2x + 5y - 4z = -16 \quad E_2$
$x + y - z = -4 \quad E_3$

Multiply E_1 by 2 and add to E_2 to eliminate x. Also add E_1 to E_3 to eliminate x.

$$-2x + 4y - 2z = -8 \quad 2E_1$$
$$2x + 5y - 4z = -16 \quad E_2$$
$$9y - 6z = -24 \quad E_4$$
$$-x + 2y - z = -4 \quad E_1$$
$$x + y - z = -4 \quad E_3$$
$$3y - 2z = -8 \quad E_5$$

Equivalent system:

$$-x + 2y - z = -4 \quad E_1$$
$$9y - 6z = -24 \quad E_4$$
$$3y - 2z = -8 \quad E_5$$

If E_5 is multiplied by –3 and added to E_4, 0 = 0 results. The system is dependent and equivalent to

$$-x + 2y - z = -4$$
$$3y - 2z = -8$$

Let $z = s$. Then

$$3y - 2s = -8$$
$$y = \frac{2s-8}{3} \text{ or } \frac{2}{3}s - \frac{8}{3}$$
$$-x + 2\left(\frac{2}{3}s - \frac{8}{3}\right) - s = -4$$
$$-x = -\frac{4}{3}s + \frac{16}{3} + s - 4$$
$$x = \frac{1}{3}s - \frac{4}{3}$$

Solutions: $\left\{\left(\frac{1}{3}s - \frac{4}{3}, \frac{2}{3}s - \frac{8}{3}, s\right) \middle| s \text{ is any real number}\right\}$

47. $x = 2 + p - 2q$
$y = 3 - p + 3q$

Solve the first equation for p in terms of q, x, and y and substitute into the second equation to eliminate p, then solve for q in terms of x and y.

$$p = x - 2 + 2q$$
$$y = 3 - (x - 2 + 2q) + 3q$$
$$y = 3 - x + 2 - 2q + 3q$$
$$y = 5 - x + q$$
$$q = x + y - 5$$

Now substitute this expression for q into $p = x - 2 + 2q$ to find p in terms of x and y.

$p = x - 2 + 2(x + y - 5)$
$p = x - 2 + 2x + 2y - 10$
$p = 3x + 2y - 12$ Solution: $p = 3x + 2y - 12$, $q = x + y - 5$

To check this solution substitute into the original equations to see if true statements result:

$x = 2 + p - 2q$
$x \stackrel{?}{=} 2 + (3x + 2y - 12) - 2(x + y - 5)$
$x \stackrel{?}{=} 2 + 3x + 2y - 12 - 2x - 2y + 10$
$x \stackrel{\sqrt{}}{=} x$

$y = 3 - p + 3q$
$y \stackrel{?}{=} 3 - (3x + 2y - 12) + 3(x + y - 5)$
$y \stackrel{?}{=} 3 - 3x - 2y + 12 + 3x + 3y - 15$
$y \stackrel{\sqrt{}}{=} y$

49. $ax + by = h$
$cx + dy = k$

Solve the first equation for x in terms of y and the constants.

$ax = h - by$

$x = \frac{h - by}{a}$ $(a \neq 0)$

Substitute this expression into the second equation to eliminate x.

$$c\left(\frac{h-by}{a}\right) + dy = k$$
$$ac\left(\frac{h-by}{a}\right) + ady = ak$$
$$c(h - by) + ady = ak$$
$$ch - bcy + ady = ak$$
$$(ad - bc)y = ak - ch$$
$$y = \frac{ak - ch}{ad - bc}$$
$$ad - bc \neq 0$$

Similarly, solve the first equation for y in terms of x and the constants.

$by = h - ax$

$y = \frac{h - ax}{b}$ $(b \neq 0)$

Substitute this expression into the second equation to eliminate y.

$$cx + d\left(\frac{h-ax}{b}\right) = k$$
$$bcx + bd\left(\frac{h-ax}{b}\right) = bk$$
$$bcx + d(h - ax) = bk$$
$$bcx + dh - adx = bk$$
$$(bc - ad)x = bk - dh$$
$$x = \frac{bk - dh}{bc - ad} \quad bc - ad \neq 0$$

or, for consistency with the expression for y, $x = \frac{dh - bk}{ad - bc}$

Solution: $x = \frac{dh - bk}{ad - bc}$, $y = \frac{ak - ch}{ad - bc}$ $ad - bc \neq 0$

51. Let x = airspeed of the plane
y = rate at which wind is blowing

Then $x - y$ = ground speed flying from Atlanta to Los Angeles (head wind)
$x + y$ = ground speed flying from Los Angeles to Atlanta (tail wind)

Then, applying Distance = Rate × Time, we have

$2{,}100 = 8.75(x - y)$
$2{,}100 = 5(x + y)$

After simplification, we have

$x - y = 240$
$x + y = 420$

Solve the first equation for x in terms of y and substitute into the second equation. $x = 240 + y$

$240 + y + y = 420$
$2y = 180$
$y = 90$ mph = wind rate
$x = 240 + y$
$x = 240 + 90$
$x = 330$ mph = airspeed

53. Let x = time rowed upstream

55. Let x = amount of first batch

y = time rowed downstream

Then $x + y = \frac{1}{4}$ (15 min = $\frac{1}{4}$ hr.)

Since rate upstream = 20 – 2 = 18 mph and
rate downstream = 20 + 2 = 22 mph,

applying Distance = Rate × Time to the equal distances upstream and downstream, we have

$18x = 22y$

Solve the first equation for y in terms of x and substitute into the second equation.

$$y = \frac{1}{4} - x$$

$$18x = 22\left(\frac{1}{4} - x\right)$$

$$18x = 5.5 - 22x$$

$$40x = 5.5$$

$$x = 0.1375 \text{ hr.}$$

Then the distance rowed upstream

$= 18x = 18(0.1375) = 2.475$ km.

y = amount of second batch

Then the amount of dark chocolate in any mix of these will be $0.5x + 0.8y$, hence in 100 pounds of a 68% dark chocolate mix

$0.5x + 0.8y = 0.68(100)$

Also, the amount of milk chocolate in any mix of these will be $0.5x + 0.2y$, hence since 100 – 68 = 32 percent

$0.5x + 0.2y = 0.32(100)$

For convenience, eliminate decimals by multiplying both sides of both equations by 10.

$$5x + 8y = 680$$
$$5x + 2y = 320$$

If we multiply the second equation by –1 and add, we can eliminate x.

$$\begin{aligned} 5x + 8y &= 680 \\ \underline{-5x - 2y} &= \underline{-320} \\ 6y &= 360 \\ y &= 60 \end{aligned}$$

Since there are 100 pounds in the mix, clearly $x = 40$.

$x = 40$ lbs. of 50–50 mix, $y = 60$ lbs. of 80–20 mix.

57. "Break even" means Cost = Revenue.

Let y = Cost = Revenue.

Let x = number of CDs sold

y = Revenue = number of CDs sold × price per CD

$y = x(8.00)$

y = Cost = Fixed Cost + Variable Cost = 17,680 + number of CDs × cost per CD

$y = 17{,}680 + x(4.60)$

Substitute y from the first equation into the second equation to eliminate y.

$$8.00x = 17{,}680 + 4.60x$$

$$3.40x = 17{,}680$$

$$x = \frac{17{,}680}{3.40}$$

$x = 5{,}200$ CDs

59. Let x = number of hours Mexico plant is operated

y = number of hours Taiwan plant is operated

Then (Production at Mexico plant)	+	(Production at Taiwan plant)	=	(Total Production)
$40x$	+	$20y$	=	4000 (keyboards)
$32x$	+	$32y$	=	4000 (screens)

Solve the first equation for y in terms of x and substitute into the second equation.

$$20y = 4{,}000 - 40x$$

$$y = 200 - 2x$$

$$32x + 32(200 - 2x) = 4{,}000$$

$$32x + 6{,}400 - 64x = 4{,}000$$

$$-32x = -2{,}400$$

$x = 75$ hours Mexico plant

$y = 200 - 2x = 200 - 2(75) = 50$ hours Taiwan plant

61. (A) If $p = 4$, then $4 = 0.007q + 3$, $q = \frac{1}{0.007} = 143$ T–shirts is the number that suppliers are willing to supply at this price.

$4 = -0.018q + 15$, $q = \dfrac{11}{0.018} = 611$ T–shirts is the number that consumers will purchase.

Demand exceeds supply and the price will rise.

(B) If $p = 8$, then $8 = 0.007q + 3$, $q = \dfrac{5}{0.007} = 714$ T–shirts is the number that suppliers are willing to supply.

$8 = -0.018q + 15$, $q = \dfrac{7}{0.018} = 389$ T–shirts is the number that consumers will purchase at this price.

Supply exceeds demand and the price will fall.

(C) Solve $p = 0.007q + 3$

$p = -0.018q + 15$

Substitute the expression for p in terms of q from the first equation into the second equation.

$0.007q + 3 = -0.018q + 15$

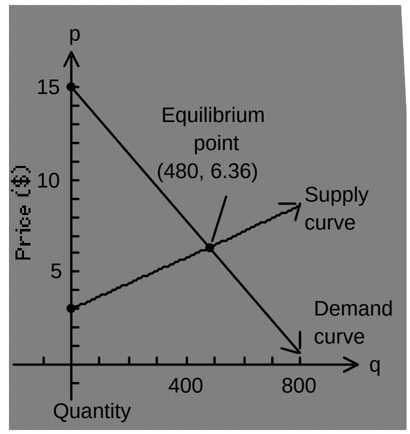
$0.025q + 3 = 15$

$0.025q = 12$

$q = 480$ T–shirts is the equilibrium quantity.

$p = 0.007(480) + 3 = \$6.36$ is the equilibrium price.

(D)



63. (A) Write $p = aq + b$.

Since $p = 0.60$ corresponds to supply $q = 450$, $0.60 = 450a + b$

Since $p = 0.90$ corresponds to supply $q = 750$, $0.90 = 750a + b$

Solve the first equation for b in terms of a and substitute into the second equation.

$b = 0.60 - 450a$

$0.90 = 750a + 0.60 - 450a$

$0.30 = 300a$

$a = 0.001$

$b = 0.60 - 450a = 0.60 - 450(0.001) = 0.15$

Thus, the supply equation is $p = 0.001q + 0.15$.

(B) Write $p = cq + d$.

Since $p = 0.60$ corresponds to demand $q = 645$, $0.60 = 645c + d$

Since $p = 0.90$ corresponds to demand $q = 495$, $0.90 = 495c + d$

Solve the first equation for d in terms of c and substitute into the second equation.

$d = 0.60 - 645c$

$0.90 = 495c + 0.60 - 645c$

$0.30 = -150c$

$c = -0.002$

$d = 0.60 - 645c = 0.60 - 645(-0.002) = 1.89$

Thus, the demand equation is $p = -0.002q + 1.89$.

(C) Solve the system of equations

$p = 0.001q + 0.15$

$p = -0.002q + 1.89$

Substitute p from the first equation into the second equation to eliminate p.

$0.001q + 0.15 = -0.002q + 1.89$

$0.003q = 1.74$

$q = 580$ bushels = equilibrium quantity

$p = 0.001q + 0.15 = 0.001(580) + 0.15 = \$0.73$ equilibrium price

65. Let p = time of primary wave

s = time for secondary wave

We know $s - p = 16$ (time difference)

To find a second equation, we have to use Distance = Rate × Time

$5p$ = distance for primary wave
$3s$ = distance for secondary wave
These distances are equal, hence $5p = 3s$
Solve the first equation for s in terms of p and substitute into the second equation to eliminate s.

$$s = p + 16$$
$$5p = 3(p + 16)$$
$$5p = 3p + 48$$
$$2p = 48$$
$$p = 24 \text{ seconds}$$
$$s = 24 + 16$$
$$s = 40 \text{ seconds}$$

The distance traveled = $5p = 3s = 120$ miles

67. Let x = number of lawn mowers manufactured each week
y = number of snowblowers manufactured each week
z = number of chain saws manufactured each week
Then

E_1 $20x + 30y + 45z = 35{,}000$ Labor
E_2 $35x + 50y + 40z = 50{,}000$ Materials
E_3 $15x + 25y + 10z = 20{,}000$ Shipping

Multiply E_3 by –4.5 and add to E_1 to eliminate z. Also multiply E_3 by –4 and add to E_2 to eliminate z.

$$\begin{array}{ll} 20x + 30y + 45z = 35{,}000 & E_1 \\ \underline{-67.5x - 112.5y - 45z = -90{,}000} & (-4.5)E_3 \\ -47.5x - 82.5y = -55{,}000 & E_4 \\ 35x + 50y + 40z = 50{,}000 & E_2 \\ \underline{-60x - 100y - 40z = -80{,}000} & (-4)E_3 \\ -25x - 50y = -30{,}000 & E_5 \end{array}$$

Equivalent system:

$$\begin{array}{ll} 15x + 25y + 10z = 20{,}000 & E_3 \\ -47.5x - 82.5y = -55{,}000 & E_4 \\ -25x - 50y = -30{,}000 & E_5 \end{array}$$

Multiply E_5 by –1.9 and add to E_4 to eliminate x.

$$\begin{array}{ll} 47.5x + 95y = 57{,}000 & (-1.9)E_5 \\ \underline{-47.5x - 82.5y = -55{,}000} & E_4 \\ 12.5y = 2{,}000 & \\ y = 160 & \end{array}$$

Substitute $y = 160$ into E_5 and solve for x.

$$\begin{array}{ll} -25x - 50y = -30{,}000 & E_5 \\ -25x - 50(160) = -30{,}000 & \\ x = 880 & \end{array}$$

Substitute $x = 880$ and $y = 160$ into E_3 and solve for z.

$$\begin{array}{ll} 15x + 25y + 10z = 20{,}000 & E_3 \\ 15(880) + 25(160) + 10z = 20{,}000 & \\ z = 280 & \end{array}$$

880 lawn mowers, 160 snow blowers, 280 chain saws.

69. Let x = number of days operating the Michigan plant
y = number of days operating the New York plant
z = number of days operating the Ohio plant
Then

E_1 $10x + 70y + 60z = 2{,}150$ Notebooks

E_2 $20x + 50y + 80z = 2,300$ Desktops
E_3 $40x + 30y + 90z = 2,500$ Servers

Multiply E_1 by –2 and add to E_2 to eliminate x. Also multiply E_1 by –4 and add to E_3 to eliminate x.

$$
\begin{array}{rll}
-20x - 140y - 120z = -4,300 & & (-2)E_1 \\
\underline{20x + 50y + 80z = 2,300} & & E_2 \\
-90y - 40z = -2,000 & & E_4 \\
-40x - 280y - 240z = -8,600 & & (-4)E_1 \\
\underline{40x + 30y + 90z = 2,500} & & E_3 \\
-250y - 150z = -6,100 & & E_5
\end{array}
$$

Equivalent system:

$$
\begin{array}{rl}
10x + 70y + 60z = 2,150 & E_1 \\
-90y - 40z = -2,000 & E_4 \\
-250y - 150z = -6,100 & E_5
\end{array}
$$

Multiply E_5 by –0.36 and add to E_4 to eliminate y.

$$
\begin{array}{rl}
-90y - 40z = -2,000 & E_4 \\
\underline{90y + 54z = 2,196} & (0.36)E_5 \\
14z = 196 & \\
z = 14 &
\end{array}
$$

Substitute $z = 14$ into E_4 and solve for y.

$$
\begin{array}{rl}
-90y - 40z = -2,000 & E_4 \\
-90y - 40(14) = -2,000 & \\
y = 16 &
\end{array}
$$

Substitute $y = 16$ and $z = 14$ into E_1 and solve for x.

$$
\begin{array}{rl}
10x + 70y + 60z = 2,150 & E_1 \\
10x + 70(16) + 60(14) = 2,150 & \\
x = 19 &
\end{array}
$$

19 days Michigan plant, 16 days New York plant, 14 days Ohio plant

71. Let

x = amount invested in treasury bonds at 4%
y = amount invested in municipal bonds at 3.5%
z = amount invested in corporate bonds at 4.5%

Then

E_1 $x + y + z = 70,000$ total investment

E_2 $0.04x + 0.035y + 0.045z = 2{,}900$ interest income

E_3 $x = y + z$ tax considerations

Multiply E_2 by 1,000 and rewrite E_3 to obtain:

$$x + y + z = 70{,}000 \quad E_1$$
$$40x + 35y + 45z = 2{,}900{,}000 \quad E_4$$
$$x - y - z = 0 \quad E_5$$

Add E_1 and E_5 to eliminate y and z.

$$x + y + z = 70{,}000 \quad E_1$$
$$x - y - z = 0 \quad E_5$$
$$2x = 70{,}000$$
$$x = 35{,}000$$

Substitute $x = 35{,}000$ into E_1 and E_4

$$35{,}000 + y + z = 70{,}000$$
$$40(35{,}000) + 35y + 45z = 2{,}900{,}000$$

Simplify to obtain

$$y + z = 35{,}000 \quad E_6$$
$$35y + 45z = 1{,}500{,}000 \quad E_7$$

Multiply E_6 by –35 and add to E_7 to eliminate y.

$$-35y - 35z = -1{,}225{,}000 \quad (-35)E_6$$
$$35y + 45z = 1{,}500{,}000 \quad E_7$$
$$10z = 275{,}000$$
$$z = 27{,}500$$

Substitute $z = 27{,}500$ into E_6 and solve for y.

$$y + 27{,}500 = 35{,}000$$
$$y = 7{,}500$$

$35,000 treasury bonds, $7,500 municipal bonds, $27,500 corporate bonds.

Section 10–2

1. The size of a matrix is given by $m \times n$, where m is the number of rows and n is the number of columns.

3. A column matrix is a matrix with only one column. Its size is given by $m \times 1$, where m is the number of rows.

5. a_{ij} is the element in row i, column j of a matrix.

7. The augmented coefficient matrix of a system of equations is the coefficient matrix with one added column, the column of constants.

9. The reduced matrix of a system is a matrix row–equivalent to the augmented coefficient matrix, from which the solutions of the system can be directly read off.

11. No. Condition 2 is violated. **13.** Yes **15.** No. Condition 4 is violated. **17.** Yes

19.
$$x_1 = -2$$
$$x_2 = 3$$
$$x_3 = 0$$
The system is already solved.

21.
$$x_1 - 2x_3 = 3$$
$$x_2 + x_3 = -5$$

Solution:

$x_3 = t$

$x_2 = -5 - x_3 = -5 - t$

$x_1 = 3 + 2x_3 = 3 + 2t$

23.
$$x_1 = 0$$
$$x_2 = 0$$
$$0 = 1$$
The system has no solution.

Thus $x_1 = 2t + 3$, $x_2 = -t - 5$, $x_3 = t$ is the solution for t any real number.

25. $x_1 - 2x_2 \quad - 3x_4 = -5$

$x_3 + 3x_4 = 2$

Solution:

$x_4 = t$

$x_3 = 2 - 3x_4 = 2 - 3t$

$x_2 = s$

$x_1 = -5 + 2x_2 + 3x_4 = -5 + 2s + 3t$

Thus $x_1 = 2s + 3t - 5$, $x_2 = s$, $x_3 = -3t + 2$, $x_4 = t$ is the solution, for s and t any real numbers.

27. $R_1 \leftrightarrow R_2$ means interchange Rows 1 and 2.

$$\left[\begin{array}{cc|c} 4 & -6 & -8 \\ 1 & -3 & 2 \end{array}\right]$$

29. $-4R_1 \to R_1$ means multiply Row 1 by −4.

$$\left[\begin{array}{cc|c} -4 & 12 & -8 \\ 4 & -6 & -8 \end{array}\right]$$

31. $2R_2 \to R_2$ means multiply Row 2 by 2.

$$\left[\begin{array}{cc|c} 1 & -3 & 2 \\ 8 & -12 & -16 \end{array}\right]$$

33. $(-4)R_1 + R_2 \to R_2$ means replace Row 2 by itself plus −4 times Row 1.

6 44 7 4 48

$$\left[\begin{array}{cc|c} 1 & -3 & 2 \\ 4 & -6 & -8 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & 2 \\ 0 & 6 & -16 \end{array}\right]$$

−4 12 −8

35. $(-2)R_1 + R_2 \to R_2$ means replace Row 2 by itself plus −2 times Row 1.

6 44 7 4 48

$$\left[\begin{array}{cc|c} 1 & -3 & 2 \\ 4 & -6 & -8 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & 2 \\ 2 & 0 & -12 \end{array}\right]$$

−2 6 −4

37. $(-1)R_1 + R_2 \to R_2$ means replace Row 2 by itself plus −1 times Row 1.

6 44 7 4 48

$$\left[\begin{array}{cc|c} 1 & -3 & 2 \\ 4 & -6 & -8 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & 2 \\ 3 & -3 & -10 \end{array}\right]$$

−1 3 −2

39. $$\left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & 1 & 3 \end{array}\right] \quad (-2)R_2 + R_1 \to R_1$$

Need a 0 here

$$\sim \left[\begin{array}{cc|c} 1 & 0 & -7 \\ 0 & 1 & 3 \end{array}\right]$$

41. $$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 3 & -6 \end{array}\right] \tfrac{1}{3}R_3 \to R_3$$

↑

Need a 1 here

Need 0's here

↓

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{l} 3R_3 + R_1 \to R_1 \\ (-2)R_3 + R_2 \to R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

43. $$\left[\begin{array}{ccc|c} 1 & 2 & -2 & -1 \\ 0 & 3 & -6 & 1 \\ 0 & -1 & 2 & -\tfrac{1}{3} \end{array}\right] \tfrac{1}{3}R_2 \to R_2$$

Need a 1 here

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -2 & -1 \\ 0 & 1 & -2 & \tfrac{1}{3} \\ 0 & -1 & 2 & -\tfrac{1}{3} \end{array}\right] \begin{array}{l} (-2)R_2 + R_1 \to R_1 \\ \\ R_3 + R_2 \to R_3 \end{array}$$

45. $$\left[\begin{array}{cc|c} 1 & -4 & -2 \\ -2 & 1 & -3 \end{array}\right] \quad 2R_1 + R_2 \to R_2$$

↑

Need a 0 here

2 −8 −4

$$\sim \left[\begin{array}{cc|c} 1 & -4 & -2 \\ 0 & -7 & -7 \end{array}\right] \quad -\tfrac{1}{7}R_2 \to R_2$$

↑

Need 0's here

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & 2 & -\frac{5}{3} \\ 0 & 1 & -2 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Need a 1 here
Need a 0 here

$$\sim\left[\begin{array}{cc|c} 1 & -4 & -2 \\ 0 & 1 & 1 \end{array}\right] \quad 4R_2 + R_1 \to R_1$$

0 4 4

$$\sim\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

Therefore $x_1 = 2$ and $x_2 = 1$

47. $\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & 4 & -8 \end{array}\right] \quad (-2)R_1 + R_2 \to R_2$ 1 2 3

Need a 0 here

−2 −4 −8

$$\sim\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 0 & -16 \end{array}\right]$$

This matrix corresponds to the system

$x_1 + 2x_2 = 4$

$0x_1 + 0x_2 = -16$

This system has no solution.

49. $\left[\begin{array}{cc|c} 3 & -6 & -9 \\ -2 & 4 & 6 \end{array}\right] \quad \frac{1}{3}R_1 \to R_1$

Need a 1 here

$$\sim\left[\begin{array}{cc|c} 1 & -2 & -3 \\ -2 & 4 & 6 \end{array}\right] \quad -2R_1 + R_2 \to R_2$$

Need a 0 here

2 −4 −6

$$\sim\left[\begin{array}{cc|c} 1 & -2 & -3 \\ 0 & 0 & 0 \end{array}\right]$$

This matrix corresponds to the system

$x_1 - 2x_2 = -3$

$0x_1 + 0x_2 = 0$

Thus $x_1 = 2x_2 - 3$.

Hence there are infinitely many solutions: for any real number s, $x_2 = s$, $x_1 = 2s - 3$ is a solution.

51. $\left[\begin{array}{ccc|c} 2 & 4 & -10 & -2 \\ 3 & 9 & -21 & 0 \\ 1 & 5 & -12 & 1 \end{array}\right] \quad R_1 \leftrightarrow R_3$

Need a 1 here

$$\sim\left[\begin{array}{ccc|c} 1 & 5 & -12 & 1 \\ 3 & 9 & -21 & 0 \\ 2 & 4 & -10 & -2 \end{array}\right] \quad \begin{array}{l} (-3)R_1 + R_2 \to R_2 \\ (-2)R_1 + R_3 \to R_3 \end{array}$$

Need 0's here

$$\sim\left[\begin{array}{ccc|c} 1 & 5 & -12 & 1 \\ 0 & -6 & 15 & -3 \\ 0 & -6 & 14 & -4 \end{array}\right] \quad -\frac{1}{6}R_2 \to R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 1 & -\frac{5}{2} & \frac{1}{2} \\ 0 & 0 & -1 & -1 \end{array}\right] \quad -R_3 \to R_3$$

Need a 1 here

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 1 & -\frac{5}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} (-\frac{1}{2})R_3 + R_1 \to R_1 \\ \frac{5}{2}R_3 + R_2 \to R_2 \end{array}$$

Need 0's here

Need a 1 here

$$\sim\left[\begin{array}{ccc|c} 1 & 5 & -12 & 1 \\ 0 & 1 & -\frac{5}{2} & \frac{1}{2} \\ 0 & -6 & 14 & -4 \end{array}\right] \begin{array}{l} (-5)R_2 + R_1 \to R_1 \\ \\ 6R_2 + R_3 \to R_3 \end{array}$$

Need 0's here

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Therefore $x_1 = -2$, $x_2 = 3$, and $x_3 = 1$.

53. $$\left[\begin{array}{ccc|c} 3 & 8 & -1 & -18 \\ 2 & 1 & 5 & 8 \\ 2 & 4 & 2 & -4 \end{array}\right] \begin{array}{l} \frac{1}{2}R_3 \to R_3 \\ \\ R_3 \leftrightarrow R_1 \end{array}$$

Need a 1 here

$$\sim\left[\begin{array}{ccc|c} 1 & 2 & 1 & -2 \\ 2 & 1 & 5 & 8 \\ 3 & 8 & -1 & -18 \end{array}\right] \begin{array}{l} \\ (-2)R_1 + R_2 \to R_2 \\ (-3)R_1 + R_3 \to R_3 \end{array}$$

Need 0's here

$$\sim\left[\begin{array}{ccc|c} 1 & 2 & 1 & -2 \\ 0 & -3 & 3 & 12 \\ 0 & 2 & -4 & -12 \end{array}\right] -\tfrac{1}{3}R_2 \to R_2$$

Need a 1 here

$$\sim\left[\begin{array}{ccc|c} 1 & 2 & 1 & -2 \\ 0 & 1 & -1 & -4 \\ 0 & 2 & -4 & -12 \end{array}\right] \begin{array}{l} (-2)R_2 + R_1 \to R_1 \\ \\ (-2)R_2 + R_3 \to R_3 \end{array}$$

Need 0's here

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & 3 & 6 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & -2 & -4 \end{array}\right] \begin{array}{l} \\ \\ -\frac{1}{2}R_3 \to R_3 \end{array}$$

Need a 1 here

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & 3 & 6 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} (-3)R_3 + R_1 \to R_1 \\ R_3 + R_2 \to R_2 \\ \end{array}$$

Need 0's here

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Therefore $x_1 = 0$, $x_2 = -2$, and $x_3 = 2$.

55. $$\left[\begin{array}{ccc|c} 2 & -1 & -3 & 8 \\ 1 & -2 & 0 & 7 \end{array}\right] R_1 \leftrightarrow R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & -2 & 0 & 7 \\ 2 & -1 & -3 & 8 \end{array}\right] (-2)R_1 + R_2 \to R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & -2 & 0 & 7 \\ 0 & 3 & -3 & -6 \end{array}\right] \tfrac{1}{3}R_2 \to R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & -2 & 0 & 7 \\ 0 & 1 & -1 & -2 \end{array}\right] 2R_2 + R_1 \to R_1 \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & -1 & -2 \end{array}\right]$$

Let $x_3 = t$. Then

$x_2 - x_3 = -2$

$x_2 = x_3 - 2 = t - 2$

$x_1 - 2x_3 = 3$

57. $$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 3 & 2 & 7 \\ 1 & -1 & -1 \end{array}\right] R_1 \leftrightarrow R_3$$

$$\sim\left[\begin{array}{cc|c} 1 & -1 & -1 \\ 3 & 2 & 7 \\ 2 & -1 & 0 \end{array}\right] \begin{array}{l} \\ (-3)R_1 + R_2 \to R_2 \\ (-2)R_1 + R_3 \to R_3 \end{array}$$

$$\sim\left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 5 & 10 \\ 0 & 1 & 2 \end{array}\right] R_2 \leftrightarrow R_3$$

$x_1 = 2x_3 + 3 = 2t + 3$

Solution: $x_1 = 2t + 3$, $x_2 = t - 2$, $x_3 = t$, t any real number.

$$\sim\left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 5 & 10 \end{array}\right] \begin{array}{l} R_2 + R_1 \to R_1 \\ \\ (-5)R_2 + R_3 \to R_3 \end{array} \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore $x_1 = 1$ and $x_2 = 2$.

59. $$\left[\begin{array}{ccc|c} 3 & -4 & -1 & 1 \\ 2 & -3 & 1 & 1 \\ 1 & -2 & 3 & 2 \end{array}\right] R_1 \leftrightarrow R_3$$

$$\sim\left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 2 & -3 & 1 & 1 \\ 3 & -4 & -1 & 1 \end{array}\right] \begin{array}{l} \\ (-2)R_1 + R_2 \to R_2 \\ (-3)R_1 + R_3 \to R_3 \end{array}$$

$$\sim\left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 1 & -5 & -3 \\ 0 & 2 & -10 & -5 \end{array}\right] \begin{array}{l} \\ \\ (-2)R_2 + R_3 \to R_3 \end{array}$$

$$\sim\left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Since the last row corresponds to the equation $0x_1 + 0x_2 + 0x_3 = 1$, there is no solution.

61. $$\left[\begin{array}{ccc|c} 2 & -2 & -4 & -2 \\ -3 & 3 & 6 & 3 \end{array}\right] \begin{array}{l} \frac{1}{2}R_1 \to R_1 \\ \frac{1}{3}R_2 \to R_2 \end{array}$$

$$\sim\left[\begin{array}{ccc|c} 1 & -1 & -2 & -1 \\ -1 & 1 & 2 & 1 \end{array}\right] R_1 + R_2 \to R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_3 = t$, $x_2 = s$. Then

$x_1 - x_2 - 2x_3 = -1$

$x_1 = x_2 + 2x_3 - 1$

$= s + 2t - 1$

Solution: $x_1 = s + 2t - 1$, $x_2 = s$, $x_3 = t$, s and t any real numbers.

63. $$\left[\begin{array}{ccc|c} 2 & -5 & -3 & 7 \\ -4 & 10 & 2 & 6 \\ 6 & -15 & -1 & -19 \end{array}\right] \begin{array}{l} \\ 2R_1 + R_2 \to R_2 \\ (-3)R_1 + R_3 \to R_3 \end{array}$$

$$\sim\left[\begin{array}{ccc|c} 2 & -5 & -3 & 7 \\ 0 & 0 & -4 & 20 \\ 0 & 0 & 8 & -40 \end{array}\right] \begin{array}{l} \frac{1}{2}R_1 \to R_1 \\ -\frac{1}{4}R_2 \to R_2 \\ \end{array}$$

$$\sim\left[\begin{array}{ccc|c} 1 & -2.5 & -1.5 & 3.5 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 8 & -40 \end{array}\right] \begin{array}{l} 1.5R_2 + R_1 \to R_1 \\ \\ (-8)R_2 + R_3 \to R_3 \end{array}$$

$$\sim\left[\begin{array}{ccc|c} 1 & -2.5 & 0 & -4 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_2 = t$. Then $x_3 = -5$ and

$x_1 - 2.5x_2 = -4$

$x_1 = 2.5x_2 - 4$

$x_1 = 2.5t - 4$

Solution: $x_1 = 2.5t - 4$, $x_2 = t$, $x_3 = -5$, t any real number.

65. $$\left[\begin{array}{cccc|c} 1 & 2 & -4 & -1 & 7 \\ 2 & 5 & -9 & -4 & 16 \\ 1 & 5 & -7 & -7 & 13 \end{array}\right] \begin{array}{l} \\ (-2)R_1 + R_2 \to R_2 \\ (-1)R_1 + R_3 \to R_3 \end{array}$$

$$\sim\left[\begin{array}{cccc|c} 1 & 2 & -4 & -1 & 7 \\ 0 & 1 & -1 & -2 & 2 \\ 0 & 3 & -3 & -6 & 6 \end{array}\right] \begin{array}{l} (-2)R_2 + R_1 \to R_1 \\ \\ (-3)R_2 + R_3 \to R_3 \end{array}$$

$$\sim\left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & 3 \\ 0 & 1 & -1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_4 = t$, $x_3 = s$. Then

$x_2 - x_3 - 2x_4 = 2$

$x_2 = s + 2t + 2$

$x_1 - 2x_3 + 3x_4 = 3$

$x_1 = 2s - 3t + 3$

Solution: $x_1 = 2s - 3t + 3$, $x_2 = s + 2t + 2$, $x_3 = s$, $x_4 = t$, s and t any real numbers.

67. $\left[\begin{array}{rrrr|r} 1 & -1 & 3 & -2 & 1 \\ -2 & 4 & -3 & 1 & 0.5 \\ 3 & -1 & 10 & -4 & 2.9 \\ 4 & -3 & 8 & -2 & 0.6 \end{array}\right] \begin{array}{l} \\ 2R_1 + R_2 \to R_2 \\ (-3)R_1 + R_3 \to R_3 \\ (-4)R_1 + R_4 \to R_4 \end{array} \sim \left[\begin{array}{rrrr|r} 1 & -1 & 3 & -2 & 1 \\ 0 & 2 & 3 & -3 & 2.5 \\ 0 & 2 & 1 & 2 & -0.1 \\ 0 & 1 & -4 & 6 & -3.4 \end{array}\right] R_4 \leftrightarrow R_2$

$\sim \left[\begin{array}{rrrr|r} 1 & -1 & 3 & -2 & 1 \\ 0 & 1 & -4 & 6 & -3.4 \\ 0 & 2 & 1 & 2 & -0.1 \\ 0 & 2 & 3 & -3 & 2.5 \end{array}\right] \begin{array}{l} R_2 + R_1 \to R_1 \\ \\ (-2)R_2 + R_3 \to R_3 \\ (-2)R_2 + R_4 \to R_4 \end{array} \sim \left[\begin{array}{rrrr|r} 1 & 0 & -1 & 4 & -2.4 \\ 0 & 1 & -4 & 6 & -3.4 \\ 0 & 0 & 9 & -10 & 6.7 \\ 0 & 0 & 11 & -15 & 9.3 \end{array}\right] (-1)R_4 + R_3 \to R_3$

$\sim \left[\begin{array}{rrrr|r} 1 & 0 & -1 & 4 & -2.4 \\ 0 & 1 & -4 & 6 & -3.4 \\ 0 & 0 & -2 & 5 & -2.6 \\ 0 & 0 & 11 & -15 & 9.3 \end{array}\right] -\tfrac{1}{2}R_3 \leftrightarrow R_3 \sim \left[\begin{array}{rrrr|r} 1 & 0 & -1 & 4 & -2.4 \\ 0 & 1 & -4 & 6 & -3.4 \\ 0 & 0 & 1 & -2.5 & 1.3 \\ 0 & 0 & 11 & -15 & 9.3 \end{array}\right] \begin{array}{l} R_3 + R_1 \to R_1 \\ 4R_3 + R_2 \to R_2 \\ \\ (-11)R_3 + R_4 \to R_4 \end{array}$

$\sim \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 1.5 & -1.1 \\ 0 & 1 & 0 & -4 & 1.8 \\ 0 & 0 & 1 & -2.5 & 1.3 \\ 0 & 0 & 0 & 12.5 & -5 \end{array}\right] \tfrac{1}{12.5}R_4 \to R_4 \sim \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 1.5 & -1.1 \\ 0 & 1 & 0 & -4 & 1.8 \\ 0 & 0 & 1 & -2.5 & 1.3 \\ 0 & 0 & 0 & 1 & -0.4 \end{array}\right] \begin{array}{l} (-1.5)R_4 + R_1 \to R_1 \\ 4R_4 + R_2 \to R_2 \\ 2.5R_4 + R_3 \to R_3 \\ \\ \end{array}$

$\sim \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 0 & 0.2 \\ 0 & 0 & 1 & 0 & 0.3 \\ 0 & 0 & 0 & 1 & -0.4 \end{array}\right]$

Solution: $x_1 = -0.5$, $x_2 = 0.2$, $x_3 = 0.3$, $x_4 = -0.4$

69. $\left[\begin{array}{rrrrr|r} 1 & -2 & 1 & 1 & 2 & 2 \\ -2 & 4 & 2 & 2 & -2 & 0 \\ 3 & -6 & 1 & 1 & 5 & 4 \\ -1 & 2 & 3 & 1 & 1 & 3 \end{array}\right] \begin{array}{l} \\ 2R_1 + R_2 \to R_2 \\ (-3)R_1 + R_3 \to R_3 \\ R_1 + R_4 \to R_4 \end{array} \sim \left[\begin{array}{rrrrr|r} 1 & -2 & 1 & 1 & 2 & 2 \\ 0 & 0 & 4 & 4 & 2 & 4 \\ 0 & 0 & -2 & -2 & -1 & -2 \\ 0 & 0 & 4 & 2 & 3 & 5 \end{array}\right] \tfrac{1}{4}R_2 \to R_2$

$\sim \left[\begin{array}{rrrrr|r} 1 & -2 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 0.5 & 1 \\ 0 & 0 & -2 & -2 & -1 & -2 \\ 0 & 0 & 4 & 2 & 3 & 5 \end{array}\right] \begin{array}{l} (-1)R_2 + R_1 \to R_1 \\ \\ 2R_2 + R_3 \to R_3 \\ (-4)R_2 + R_4 \to R_4 \end{array} \sim \left[\begin{array}{rrrrr|r} 1 & -2 & 0 & 0 & 1.5 & 1 \\ 0 & 0 & 1 & 1 & 0.5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right] R_3 \leftrightarrow R_4$

$\sim \left[\begin{array}{rrrrr|r} 1 & -2 & 0 & 0 & 1.5 & 1 \\ 0 & 0 & 1 & 1 & 0.5 & 1 \\ 0 & 0 & 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \left(-\tfrac{1}{2}\right)R_3 \to R_3 \sim \left[\begin{array}{rrrrr|r} 1 & -2 & 0 & 0 & 1.5 & 1 \\ 0 & 0 & 1 & 1 & 0.5 & 1 \\ 0 & 0 & 0 & 1 & -0.5 & -0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] (-1)R_3 + R_2 \to R_2$

$$\sim\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 1.5 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1.5 \\ 0 & 0 & 0 & 1 & -0.5 & -0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_5 = t$. Then

$x_4 - 0.5x_5 = -0.5$

$x_4 = 0.5x_5 - 0.5 = 0.5t - 0.5$

$x_3 + x_5 = 1.5$

$x_3 = -x_5 + 1.5 = -t + 1.5$

Let $x_2 = s$. Then

$x_1 - 2x_2 + 1.5x_5 = 1$

$x_1 = 2x_2 - 1.5x_5 + 1 = 2s - 1.5t + 1$

Solution: $x_1 = 2s - 1.5t + 1$, $x_2 = s$, $x_3 = -t + 1.5$, $x_4 = 0.5t - 0.5$, $x_5 = t$, s and t any real numbers.

71. (A) The reduced form matrix will have the form $\left[\begin{array}{ccc|c} 1 & a & b & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Thus, the system has been shown equivalent to

$x_1 + ax_2 + bx_3 = c$

$0 = 0$

$0 = 0$

The system is dependent, and x_2 and x_3 may assume any real values. Thus, there are two parameters in the solution.

(B) The reduced form matrix will have the form $\left[\begin{array}{ccc|c} 1 & 0 & a & b \\ 0 & 1 & c & d \\ 0 & 0 & 0 & 0 \end{array}\right]$

Thus, the system has been shown equivalent to

$x_1 + ax_3 = b$

$x_2 + cx_3 = d$

$0 = 0$

The system is dependent, with a solution for any real value of x_3.

Thus, there is one parameter in the solution.

(C) The reduced form matrix will have the form $\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$

Thus, there is only one solution, $x_1 = a$, $x_2 = b$, $x_3 = c$, and the system is independent.

(D) This is impossible; there are only 3 equations.

73. Let

x = the number of CD's

y = the number of DVD's

z = the number of books

Then

$x + y + z = 13$ total items

$10x + 12y + 7z = 129$ total amount spent

If we multiply the first equation by –10 and add, we can eliminate x.

$$\begin{aligned} -10x - 10y - 10z &= -130 \\ \underline{10x + 12y + 7z} &\underline{= 129} \\ 2y - 3z &= -1 \end{aligned}$$

$$2y = 3z - 1$$
$$y = \frac{3z-1}{2}$$

Since x, y, and z must be positive integers, a solution is achieved, but only for certain values of z.

If $z = 1$, $y = \frac{3\cdot 1 - 1}{2} = 1$, $x + y + z = 13$, hence $x + 1 + 1 = 13$, $x = 11$

11 CD's, 1 DVD, and 1 book

If $z = 3$, $y = \frac{3\cdot 3 - 1}{2} = 4$, $x + y + z = 13$, hence $x + 4 + 3 = 13$, $x = 6$

6 CD's, 4 DVDs, and 3 books

If $z = 5$, $y = \frac{3\cdot 5 - 1}{2} = 7$, $x + y + z = 13$, hence $x + 7 + 5 = 13$, $x = 1$

1 CD, 7 DVDs, and 5 books

No other solutions are possible.

75. Let x = amount of 20% solution
y = amount of 80% solution

Summarize the given information in a table:

	20% solution	80% solution	62% solution
amount of solution	x	y	100
amount of acid	$0.2x$	$0.8y$	$0.62(100) = 62$

Form equations from the information:

$$\begin{pmatrix}\text{Amount of}\\ \text{20\% solution}\end{pmatrix} + \begin{pmatrix}\text{Amount of}\\ \text{80\% solution}\end{pmatrix} = \begin{pmatrix}\text{Amount of}\\ \text{62\% solution}\end{pmatrix}$$
$$x + y = 100$$
$$\begin{pmatrix}\text{Amount of acid}\\ \text{in 20\% solution}\end{pmatrix} + \begin{pmatrix}\text{Amount of acid}\\ \text{in 80\% solution}\end{pmatrix} = \begin{pmatrix}\text{Amount of acid}\\ \text{in 62\% solution}\end{pmatrix}$$
$$0.2x + 0.8y = 62$$

Solve this system using elimination by addition.

$$\begin{aligned} -0.2x - 0.2y &= -20 \\ 0.2x + 0.8y &= 62 \\ \hline 0.6y &= 42 \\ y &= 70 \\ x + 70 &= 100 \\ x &= 30 \end{aligned}$$

30 liters of 20% solution and 70 liters of 80% solution.

77. If the curve passes through a point, the coordinates of the point satisfy the equation of the curve. Hence,

$3 = a + b(-2) + c(-2)^2$
$2 = a + b(-1) + c(-1)^2$
$6 = a + b(1) + c(1)^2$

After simplification, we have

$a - 2b + 4c = 3$
$a - b + c = 2$
$a + b + c = 6$

We write the augmented matrix and solve by Gauss–Jordan elimination.

$$\left[\begin{array}{rrr|r} 1 & -2 & 4 & 3 \\ 1 & -1 & 1 & 2 \\ 1 & 1 & 1 & 6 \end{array}\right] \begin{matrix} \\ (-1)R_1 + R_2 \to R_2 \\ (-1)R_1 + R_3 \to R_3 \end{matrix} \sim \left[\begin{array}{rrr|r} 1 & -2 & 4 & 3 \\ 0 & 1 & -3 & -1 \\ 0 & 3 & -3 & 3 \end{array}\right] \begin{matrix} 2R_2 + R_1 \to R_1 \\ \\ (-3)R_2 + R_3 \to R_3 \end{matrix} \sim \left[\begin{array}{rrr|r} 1 & 0 & -2 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 6 & 6 \end{array}\right] \begin{matrix} \\ \\ \tfrac{1}{6}R_3 \to R_3 \end{matrix}$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & -2 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{matrix} 2R_3 + R_1 \to R_1 \\ 3R_3 + R_2 \to R_2 \\ \\ \end{matrix} \sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Thus $a = 3$, $b = 2$, $c = 1$.

79. Let x_1 = number of one–person boats
x_2 = number of two–person boats
x_3 = number of four–person boats

We have

$0.5x_1 + 1.0x_2 + 1.5x_3 = 380$ cutting department
$0.6x_1 + 0.9x_2 + 1.2x_3 = 330$ assembly department
$0.2x_1 + 0.3x_2 + 0.5x_3 = 120$ packing department

Common Error:
The facts in this problem do not justify the equation
$0.5x_1 + 0.6x_2 + 0.2x_3 = 380$

Clearing of decimals for convenience:

$x_1 + 2x_2 + 3x_3 = 760$
$6x_1 + 9x_2 + 12x_3 = 3300$
$2x_1 + 3x_2 + 5x_3 = 1200$

We write the augmented matrix and solve by Gauss–Jordan elimination:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 760 \\ 6 & 9 & 12 & 3300 \\ 2 & 3 & 5 & 1200 \end{array}\right] \begin{array}{l} \\ (-6)R_1 + R_2 \to R_2 \\ (-2)R_1 + R_3 \to R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 760 \\ 0 & -3 & -6 & -1260 \\ 0 & -1 & -1 & -320 \end{array}\right] -\tfrac{1}{3}R_2 \to R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 760 \\ 0 & 1 & 2 & 420 \\ 0 & -1 & -1 & -320 \end{array}\right] \begin{array}{l} (-2)R_2 + R_1 \to R_1 \\ \\ R_2 + R_3 \to R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -80 \\ 0 & 1 & 2 & 420 \\ 0 & 0 & 1 & 100 \end{array}\right] \begin{array}{l} R_3 + R_1 \to R_1 \\ (-2)R_3 + R_2 \to R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & 220 \\ 0 & 0 & 1 & 100 \end{array}\right]$$

Therefore
$x_1 = 20$ one–person boats
$x_2 = 220$ two–person boats
$x_3 = 100$ four–person boats

81. This assumption discards the third equation. The system, cleared of decimals, reads

$x_1 + 2x_2 + 3x_3 = 760$
$6x_1 + 9x_2 + 12x_3 = 3300$

The augmented matrix becomes $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 760 \\ 6 & 9 & 12 & 3300 \end{array}\right]$

We solve by Gauss–Jordan elimination. We start by introducing a 0 into the lower left corner using $(-6)R_1 + R_2$ as in the previous problem:

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 760 \\ 0 & -3 & -6 & -1260 \end{array}\right] -\tfrac{1}{3}R_2 \to R_2 \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 760 \\ 0 & 1 & 2 & 420 \end{array}\right] (-2)R_2 + R_1 \to R_1 \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -80 \\ 0 & 1 & 2 & 420 \end{array}\right]$$

This augmented matrix is in reduced form. It corresponds to the system:

$x_1 - x_3 = -80$
$x_2 + 2x_3 = 420$

Let $x_3 = t$. Then

$x_2 = -2x_3 + 420$
$= -2t + 420$
$x_1 = x_3 - 80$

$= t - 80$

A solution is achieved, not for every real value of t, but for integer values of t that give rise to non–negative x_1, x_2, x_3.

$x_1 \geq 0$ means $t - 80 \geq 0$ or $t \geq 80$

$x_2 \geq 0$ means $-2t + 420 \geq 0$ or $210 \geq t$

Thus we have the solution

$x_1 = (t - 80)$ one–person boats

$x_2 = (-2t + 420)$ two–person boats

$x_3 = t$ four–person boats

$80 \leq t \leq 210$, t an integer

83. In this case we have $x_3 = 0$ from the beginning. The three equations of problem 79, cleared of decimals, read:

$$x_1 + 2x_2 = 760$$
$$6x_1 + 9x_2 = 3300$$
$$2x_1 + 3x_2 = 1200$$

The augmented matrix becomes: $\left[\begin{array}{cc|c} 1 & 2 & 760 \\ 6 & 9 & 3300 \\ 2 & 3 & 1200 \end{array}\right]$

Notice that the row operation $(-3)R_3 + R_2 \to R_2$

transforms this into the equivalent augmented matrix: $\left[\begin{array}{cc|c} 1 & 2 & 760 \\ 0 & 0 & -300 \\ 2 & 3 & 1200 \end{array}\right]$

Therefore, since the second row corresponds to the equation $0x_1 + 0x_2 = -300$ there is no solution.
No production schedule will use all the work–hours in all departments.

85. Let x_1 = number of ounces of food A.
x_2 = number of ounces of food B.
x_3 = number of ounces of food C.

Common Error:
The facts in this problem do not justify the equation
$30x_1 + 10x_2 + 10x_3 = 340$

Then

$$30x_1 + 10x_2 + 20x_3 = 340 \text{ (calcium)}$$
$$10x_1 + 10x_2 + 20x_3 = 180 \text{ (iron)}$$
$$10x_1 + 30x_2 + 20x_3 = 220 \text{ (vitamin A)}$$

or

$$3x_1 + x_2 + 2x_3 = 34$$
$$x_1 + x_2 + 2x_3 = 18$$
$$x_1 + 3x_2 + 2x_3 = 22$$

is the system to be solved. We form the augmented matrix and solve by Gauss–Jordan elimination.

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 34 \\ 1 & 1 & 2 & 18 \\ 1 & 3 & 2 & 22 \end{array}\right] R_1 \leftrightarrow R_2 \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 18 \\ 3 & 1 & 2 & 34 \\ 1 & 3 & 2 & 22 \end{array}\right] \begin{array}{l} (-3)R_1 + R_2 \to R_2 \\ (-1)R_1 + R_3 \to R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 18 \\ 0 & -2 & -4 & -20 \\ 0 & 2 & 0 & 4 \end{array}\right] -\tfrac{1}{2}R_2 \to R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 18 \\ 0 & 1 & 2 & 10 \\ 0 & 2 & 0 & 4 \end{array}\right] \begin{array}{l} (-1)R_2 + R_1 \to R_1 \\ \\ (-2)R_2 + R_3 \to R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 2 & 10 \\ 0 & 0 & -4 & -16 \end{array}\right] -\tfrac{1}{4}R_3 \to R_3 \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 2 & 10 \\ 0 & 0 & 1 & 4 \end{array}\right] (-2)R_3 + R_2 \to R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Thus
$x_1 = 8$ ounces food *A*
$x_2 = 2$ ounces food *B*
$x_3 = 4$ ounces food *C*

87. In this case we have $x_3 = 0$ from the beginning. The three equations of problem 85 become

$$30x_1 + 10x_2 = 340$$
$$10x_1 + 10x_2 = 180$$
$$10x_1 + 30x_2 = 220$$

or

$$3x_1 + x_2 = 34$$
$$x_1 + x_2 = 18$$
$$x_1 + 3x_2 = 22$$

The augmented matrix becomes $\left[\begin{array}{cc|c} 3 & 1 & 34 \\ 1 & 1 & 18 \\ 1 & 3 & 22 \end{array}\right]$

We solve by Gauss–Jordan elimination, starting by the row operation
$R_1 \leftrightarrow R_2$

$$\left[\begin{array}{cc|c} 1 & 1 & 18 \\ 3 & 1 & 34 \\ 1 & 3 & 22 \end{array}\right] \begin{array}{l} (-3)R_1 + R_2 \to R_2 \\ (-1)R_1 + R_3 \to R_3 \end{array} \sim \left[\begin{array}{cc|c} 1 & 1 & 18 \\ 0 & -2 & -20 \\ 0 & 2 & 4 \end{array}\right] R_2 + R_3 \to R_3 \sim \left[\begin{array}{cc|c} 1 & 1 & 18 \\ 0 & -2 & -20 \\ 0 & 0 & -16 \end{array}\right]$$

Since the third row corresponds to the equation
$0x_1 + 0x_2 = -16$
there is no solution.

89. In this case we discard the third equation. The system becomes

$$30x_1 + 10x_2 + 20x_3 = 340$$
$$10x_1 + 10x_2 + 20x_3 = 180$$

or

$$3x_1 + x_2 + 2x_3 = 34$$
$$x_1 + x_2 + 2x_3 = 18$$

The augmented matrix becomes $\left[\begin{array}{ccc|c} 3 & 1 & 2 & 34 \\ 1 & 1 & 2 & 18 \end{array}\right]$

We solve by Gauss–Jordan elimination, starting by the row operation $R_1 \leftrightarrow R_2$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 18 \\ 3 & 1 & 2 & 34 \end{array}\right] (-3)R_1 + R_2 \to R_2 \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 18 \\ 0 & -2 & -4 & -20 \end{array}\right] -\tfrac{1}{2}R_2 \to R_2 \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 18 \\ 0 & 1 & 2 & 10 \end{array}\right] (-1)R_2 + R_1 \to R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 2 & 10 \end{array}\right]$$

This augmented matrix is in reduced form. It corresponds to the system

$$x_1 = 8$$

$x_2 + 2x_3 = 10$
Let $x_3 = t$
Then $x_2 = -2x_3 + 10$
$= -2t + 10$

A solution is achieved, not for every real value t, but for values of t that give rise to non–negative x_2, x_3.
$x_3 \geq 0$ means $t \geq 0$
$x_2 \geq 0$ means $-2t + 10 \geq 0$, $5 \geq t$
Thus we have the solution
$x_1 = 8$ ounces food A
$x_2 = -2t + 10$ ounces food B
$x_3 = t$ ounces food C
$0 \leq t \leq 5$

91. Let x_1 = number of hours company A is to be scheduled
x_2 = number of hours company B is to be scheduled
In x_1 hours, company A can handle $30x_1$ telephone and $10x_1$ house contacts.
In x_2 hours, company B can handle $20x_2$ telephone and $20x_2$ house contacts.
We therefore have:
$30x_1 + 20x_2 = 600$ telephone contacts
$10x_1 + 20x_2 = 400$ house contacts
We form the augmented matrix and solve by Gauss–Jordan elimination.

$$\left[\begin{array}{cc|c} 30 & 20 & 600 \\ 10 & 20 & 400 \end{array}\right] \begin{array}{l} \frac{1}{10}R_1 \to R_1 \\ \frac{1}{10}R_2 \to R_2 \end{array} \sim \left[\begin{array}{cc|c} 3 & 2 & 60 \\ 1 & 2 & 40 \end{array}\right] R_1 \leftrightarrow R_2 \sim \left[\begin{array}{cc|c} 1 & 2 & 40 \\ 3 & 2 & 60 \end{array}\right] (-3)R_1 + R_2 \to R_2$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & 40 \\ 0 & -4 & -60 \end{array}\right] -\tfrac{1}{4}R_3 \to R_3 \sim \left[\begin{array}{cc|c} 1 & 2 & 40 \\ 0 & 1 & 15 \end{array}\right] (-2)R_2 + R_1 \to R_1$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 1 & 15 \end{array}\right]$$

Therefore
$x_1 = 10$ hours company A
$x_2 = 15$ hours company B

93. Let x = base price
y = surcharge for each additional pound.
Since a 5–pound package costs the base price plus 4 surcharges, $x + 4y = 27.75$
Since a 20–pound package costs the base price plus 19 surcharges, $x + 19y = 64.50$
Solve using elimination by addition.

$$\begin{aligned} -x - 4y &= -27.75 \\ x + 19y &= 64.50 \\ \hline 15y &= 36.75 \\ y &= 2.45 \end{aligned}$$

$x + 4(2.45) = 27.75$
$x = 17.95$
The base price is \$17.95 and the surcharge per pound is \$2.45.

95. Let x = number of pounds of robust blend
y = number of pounds of mild blend
Summarize the given information in a table:

	Robust blend	Mild blend
ozs. of Columbian beans	12	6
ozs. of Brazilian beans	4	10

Form equations from the information:

$$\begin{pmatrix}\text{pounds of Columbian}\\ \text{beans needed for}\\ \text{robust blend}\end{pmatrix} + \begin{pmatrix}\text{pounds of Columbian}\\ \text{beans needed for}\\ \text{mild blend}\end{pmatrix} = \begin{pmatrix}\text{Total}\\ \text{Columbian beans}\\ \text{available}\end{pmatrix}$$

$$\frac{12}{16}x + \frac{6}{16}y = 50(132)$$

$$\begin{pmatrix}\text{pounds of Brazilian}\\ \text{beans needed for}\\ \text{robust blend}\end{pmatrix} + \begin{pmatrix}\text{pounds of Brazilian}\\ \text{beans needed for}\\ \text{mild blend}\end{pmatrix} = \begin{pmatrix}\text{Total}\\ \text{Brazilian beans}\\ \text{available}\end{pmatrix}$$

$$\frac{4}{16}x + \frac{10}{16}y = 40(132)$$

Solve using elimination by addition:

$$\begin{aligned}\frac{12}{16}x + \frac{6}{16}y &= 6{,}600\\ -\frac{12}{16}x - \frac{30}{16}y &= -15{,}840\\ -\frac{24}{16}y &= -9{,}240\\ y &= 6{,}160\end{aligned}$$

$$\begin{aligned}\frac{4}{16}x + \frac{10}{16}(6{,}160) &= 40(132)\\ \frac{1}{4}x + 3{,}850 &= 5{,}280\\ \frac{1}{4}x &= 1{,}430\\ x &= 5{,}720\end{aligned}$$

5,720 pounds of the robust blend and 6,160 pounds of the mild blend.

Section 10–3

1. A and B must be the same size.

3. The number of columns of B must equal the number of rows of A, that is, if B is an $m \times n$ matrix, A must be a $n \times p$ matrix.

5. The negative of an $m \times n$ matrix A is an $m \times n$ matrix $-A$ in which each element is –1 times the corresponding element of A.

7. Multiply each element of the matrix by the number.

9. BA is an $n \times n$ matrix is which the element in row i, column j is the product of the element in row i of B times the element in column j of A.

11. $$\begin{bmatrix}5 & -2\\ 3 & 0\end{bmatrix} + \begin{bmatrix}-3 & 7\\ 1 & -6\end{bmatrix} = \begin{bmatrix}5+(-3) & (-2)+7\\ 3+1 & 0+(-6)\end{bmatrix} = \begin{bmatrix}2 & 5\\ 4 & -6\end{bmatrix}$$

13. $$\begin{bmatrix}4 & 0\\ -2 & 3\\ 8 & 1\end{bmatrix} + \begin{bmatrix}-1 & 2\\ 0 & 5\\ 4 & -6\end{bmatrix} = \begin{bmatrix}4+(-1) & 0+2\\ (-2)+0 & 3+5\\ 8+4 & 1+(-6)\end{bmatrix} = \begin{bmatrix}3 & 2\\ -2 & 8\\ 12 & -5\end{bmatrix}$$

15. These matrices have different sizes, hence the sum is not defined.

17. $\begin{bmatrix} 5 & -1 & 0 \\ 4 & 6 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 4 & -6 \\ 3 & 5 & -5 \end{bmatrix} = \begin{bmatrix} 5-2 & (-1)-4 & 0-(-6) \\ 4-3 & 6-5 & 3-(-5) \end{bmatrix} = \begin{bmatrix} 3 & -5 & 6 \\ 1 & 1 & 8 \end{bmatrix}$

19. These matrices have different sizes, hence the difference is not defined.

21. $\begin{bmatrix} 2.4 & -2.8 & 3.9 \\ -1.6 & 0 & 4.2 \end{bmatrix} - \begin{bmatrix} 7 & -2.2 & -2.2 \\ -3.2 & -3.2 & 1 \end{bmatrix} = \begin{bmatrix} 2.4-7 & (-2.8)-(-2.2) & (3.9)-(-2.2) \\ (-1.6)-(-3.2) & 0-(-3.2) & 4.2-1 \end{bmatrix} = \begin{bmatrix} -4.6 & -0.6 & 6.1 \\ 1.6 & 3.2 & 3.2 \end{bmatrix}$

23. $\begin{bmatrix} 12 & -16 & 28 \\ -8 & 36 & 20 \end{bmatrix}$

25. $[5 \ \ 3]\begin{bmatrix} 4 \\ 7 \end{bmatrix} = [5\cdot 4 + 3\cdot 7] = [41]$

27. $\begin{bmatrix} -5 \\ -3 \end{bmatrix}[4 \ \ -2] = \begin{bmatrix} (-5)4 & (-5)(-2) \\ (-3)4 & (-3)(-2) \end{bmatrix} = \begin{bmatrix} -20 & 10 \\ -12 & 6 \end{bmatrix}$

29. $[3 \ \ -2 \ \ -4]\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = [3\cdot 1 + (-2)2 + (-4)(-3)] = [11]$

31. $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}[3 \ \ -2 \ \ -4] = \begin{bmatrix} 1\cdot 3 & 1(-2) & 1(-4) \\ 2\cdot 3 & 2(-2) & 2(-4) \\ (-3)3 & (-3)(-2) & (-3)(-4) \end{bmatrix} = \begin{bmatrix} 3 & -2 & -4 \\ 6 & -4 & -8 \\ -9 & 6 & 12 \end{bmatrix}$

33. $\begin{bmatrix} -6 & 3 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} (-6)1+3\cdot 3 \\ 2\cdot 1+(-5)3 \end{bmatrix} = \begin{bmatrix} 3 \\ -13 \end{bmatrix}$

35. $\begin{bmatrix} 5 & 1 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 3 & 8 \end{bmatrix} = \begin{bmatrix} 5\cdot 2+1\cdot 3 & 5\cdot 0+1\cdot 8 \\ 4\cdot 2+6\cdot 3 & 4\cdot 0+6\cdot 8 \end{bmatrix} = \begin{bmatrix} 13 & 8 \\ 26 & 48 \end{bmatrix}$

37. $\begin{bmatrix} 8 & -3 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 8\cdot 2+(-3)0 & 8\cdot 0+(-3)6 \\ (-5)2+3\cdot 0 & (-5)0+3\cdot 6 \end{bmatrix} = \begin{bmatrix} 16 & -18 \\ -10 & 18 \end{bmatrix}$

39. C has 3 columns. A has 2 rows. Therefore, CA is not defined.

41. $BA = \begin{bmatrix} -3 & 1 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \end{bmatrix} = \begin{bmatrix} (-3)2+1\cdot 0 & (-3)(-1)+1\cdot 4 & (-3)3+1(-2) \\ 2\cdot 2+5\cdot 0 & 2(-1)+5\cdot 4 & 2\cdot 3+5(-2) \end{bmatrix} = \begin{bmatrix} -6 & 7 & -11 \\ 4 & 18 & -4 \end{bmatrix}$

43. $C^2 = \begin{bmatrix} -1 & 0 & 2 \\ 4 & -3 & 1 \\ -2 & 3 & 5 \end{bmatrix}\begin{bmatrix} -1 & 0 & 2 \\ 4 & -3 & 1 \\ -2 & 3 & 5 \end{bmatrix} = \begin{bmatrix} (-1)(-1)+0\cdot 4+2(-2) & (-1)0+0(-3)+2\cdot 3 & (-1)2+0\cdot 1+2\cdot 5 \\ 4(-1)+(-3)4+1(-2) & 4\cdot 0+(-3)(-3)+1\cdot 3 & 4\cdot 2+(-3)1+1\cdot 5 \\ (-2)(-1)+3\cdot 4+5(-2) & (-2)0+3(-3)+5\cdot 3 & (-2)2+3\cdot 1+5\cdot 5 \end{bmatrix}$

$= \begin{bmatrix} -3 & 6 & 8 \\ -18 & 12 & 10 \\ 4 & 6 & 24 \end{bmatrix}$

45. $DA = \begin{bmatrix} 3 & -2 \\ 0 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 3\cdot 2+(-2)0 & 3(-1)+(-2)4 & 3\cdot 3+(-2)(-2) \\ 0\cdot 2+(-1)0 & 0(-1)+(-1)4 & 0\cdot 3+(-1)(-2) \\ 1\cdot 2+2\cdot 0 & 1(-1)+2\cdot 4 & 1\cdot 3+2(-2) \end{bmatrix} = \begin{bmatrix} 6 & -11 & 13 \\ 0 & -4 & 2 \\ 2 & 7 & -1 \end{bmatrix}$

$C + DA = \begin{bmatrix} -1 & 0 & 2 \\ 4 & -3 & 1 \\ -2 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 6 & -11 & 13 \\ 0 & -4 & 2 \\ 2 & 7 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -11 & 15 \\ 4 & -7 & 3 \\ 0 & 10 & 4 \end{bmatrix}$

47. $0.2CD = 0.2\begin{bmatrix} -1 & 0 & 2 \\ 4 & -3 & 1 \\ -2 & 3 & 5 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} = 0.2\begin{bmatrix} (-1)3+0\cdot 0+2\cdot 1 & (-1)(-2)+0(-1)+2\cdot 2 \\ 4\cdot 3+(-3)0+1\cdot 1 & 4(-2)+(-3)(-1)+1\cdot 2 \\ (-2)3+3\cdot 0+5\cdot 1 & (-2)(-2)+3(-1)+5\cdot 2 \end{bmatrix}$

$= 0.2\begin{bmatrix} -1 & 6 \\ 13 & -3 \\ -1 & 11 \end{bmatrix} = \begin{bmatrix} -0.2 & 1.2 \\ 2.6 & -0.6 \\ -0.2 & 2.2 \end{bmatrix}$

49. $DB = \begin{bmatrix} 3 & -2 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 3(-3)+(-2)2 & 3\cdot 1+(-2)5 \\ 0(-3)+(-1)2 & 0\cdot 1+(-1)5 \\ 1(-3)+2\cdot 2 & 1\cdot 1+2\cdot 5 \end{bmatrix} = \begin{bmatrix} -13 & -7 \\ -2 & -5 \\ 1 & 11 \end{bmatrix}$

$CD = \begin{bmatrix} -1 & 6 \\ 13 & -3 \\ -1 & 11 \end{bmatrix}$ (see problem 47)

Thus, $2DB + 5CD = 2\begin{bmatrix} -13 & -7 \\ -2 & -5 \\ 1 & 11 \end{bmatrix} + 5\begin{bmatrix} -1 & 6 \\ 13 & -3 \\ -1 & 11 \end{bmatrix} = \begin{bmatrix} -26 & -14 \\ -4 & -10 \\ 2 & 22 \end{bmatrix} + \begin{bmatrix} -5 & 30 \\ 65 & -15 \\ -5 & 55 \end{bmatrix} = \begin{bmatrix} -31 & 16 \\ 61 & -25 \\ -3 & 77 \end{bmatrix}$

51. $(-1)AC$ is a matrix of size 2×3. $3DB$ is a matrix of size 3×2. Hence, $(-1)AC + 3DB$ is not defined.

53. $CD = \begin{bmatrix} -1 & 6 \\ 13 & -3 \\ -1 & 11 \end{bmatrix}$ (see problem 47)

Hence $CDA = \begin{bmatrix} -1 & 6 \\ 13 & -3 \\ -1 & 11 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \end{bmatrix}$

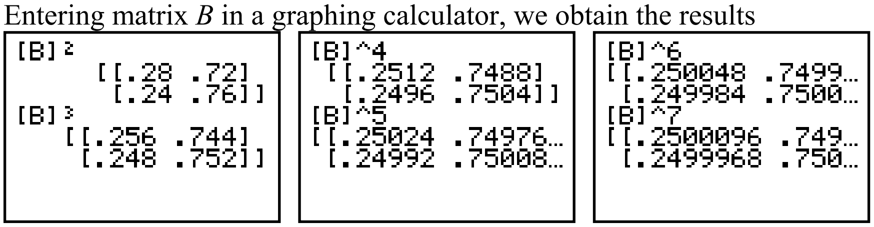
$= \begin{bmatrix} (-1)2+6\cdot 0 & (-1)(-1)+6\cdot 4 & (-1)3+6(-2) \\ 13\cdot 2+(-3)0 & 13(-1)+(-3)4 & 13\cdot 3+(-3)(-2) \\ (-1)2+11\cdot 0 & (-1)(-1)+11\cdot 4 & (-1)3+11(-2) \end{bmatrix} = \begin{bmatrix} -2 & 25 & -15 \\ 26 & -25 & 45 \\ -2 & 45 & -25 \end{bmatrix}$

55. $DB = \begin{bmatrix} -13 & -7 \\ -2 & -5 \\ 1 & 11 \end{bmatrix}$ (see problem 49)

Hence

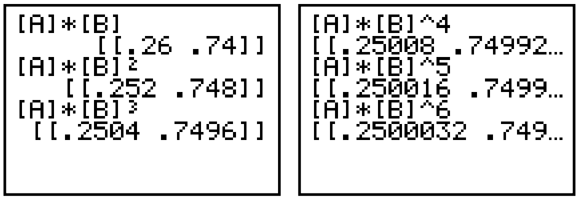
$DBA = \begin{bmatrix} -13 & -7 \\ -2 & -5 \\ 1 & 11 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \end{bmatrix} = \begin{bmatrix} (-13)2+(-7)0 & (-13)(-1)+(-7)4 & (-13)3+(-7)(-2) \\ (-2)2+(-5)0 & (-2)(-1)+(-5)4 & (-2)3+(-5)(-2) \\ 1\cdot 2+11\cdot 0 & 1(-1)+11\cdot 4 & 1\cdot 3+11(-2) \end{bmatrix} = \begin{bmatrix} -26 & -15 & -25 \\ -4 & -18 & 4 \\ 2 & 43 & -19 \end{bmatrix}$

57. Entering matrix B in a graphing calculator, we obtain the results



It appears that $B^n \to \begin{bmatrix} 0.25 & 0.75 \\ 0.25 & 0.75 \end{bmatrix}$

We calculate AB, AB^2, AB^3, … and obtain the results



It appears that $AB^n \to [0.25 \quad 0.75]$.

59. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a+2 & b-3 \\ c & d+1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$

if and only if corresponding elements are equal.

$a + 2 = 1 \quad b - 3 = -2 \quad c = 3 \quad d + 1 = -4$

$a = -1 \quad b = 1 \quad c = 3 \quad d = -5$

61. $\begin{bmatrix} 3 & 0 \\ -7 & -11 \end{bmatrix} - \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 3-w & -x \\ -7-y & -11-z \end{bmatrix} = \begin{bmatrix} 9 & 1 \\ 4 & 6 \end{bmatrix}$

if and only if corresponding elements are equal.

$3 - w = 9 \quad -x = 1 \quad -7 - y = 4 \quad -11 - z = 6$

$w = -6 \quad x = -1 \quad y = -11 \quad z = -17$

63. Compute the square matrix A:

$A^2 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+(-ab) \\ ac+(-ac) & cb+a^2 \end{bmatrix} = \begin{bmatrix} a^2+bc & 0 \\ 0 & a^2+bc \end{bmatrix}$

Two of the entries are already zero and the other two are both $a^2 + bc$. So if $a^2 + bc = 0$, then $A^2 = 0$.

If $a = 1$, $b = 1$, $c = -1$, then $a^2 + bc = 0$, so the matrix $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ will have $A^2 = 0$. If $a = 2$, $b = 4$, $c = -1$, then $a^2 + bc = 0$, so the matrix $A = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$ will have $A^2 = 0$.

65. Compute the product AB:

$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix}$

Two of the entries are $a + b$ and the other two are $c + d$, so if $a = -b$ and $c = -d$, then $AB = 0$.

The following are a couple of examples of matrices A that will satisfy $AB = 0$:

$\begin{bmatrix} 2 & -2 \\ 4 & -4 \end{bmatrix} \qquad \begin{bmatrix} -5 & 5 \\ 1 & -1 \end{bmatrix}$

67. $\begin{bmatrix} 1 & 3 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} x & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} x+9 & 7 \\ -2x-6 & -6 \end{bmatrix} = \begin{bmatrix} y & 7 \\ y & -6 \end{bmatrix}$

if and only if corresponding elements are equal.

$x + 9 = y \quad 7 \overset{\surd}{=} 7$

$-2x - 6 = y \quad -6 \overset{\surd}{=} -6$

Two conditions are already met.

To find x and y, we solve the system:

$x + 9 = y$

$-2x - 6 = y$ to obtain $x = -5$, $y = 4$. (Solution left to the student.)

69. $\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+3c & b+3d \\ a+4c & b+4d \end{bmatrix} = \begin{bmatrix} 6 & -5 \\ 7 & -7 \end{bmatrix}$

if and only if corresponding elements are equal.

$a + 3c = 6 \quad b + 3d = -5$

$a + 4c = 7 \quad b + 4d = -7$

Solving these systems we obtain $a = 3$, $b = 1$, $c = 1$, $d = -2$. (Solution left to the student.)

71. (A) Since $\begin{bmatrix} a_1 & 0 \\ 0 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & 0 \\ 0 & d_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & 0 \\ 0 & d_1+d_2 \end{bmatrix}$, the statement is true.

(B) $A + B = B + A$ is true for any matrices for which $A + B$ is defined, as it is in this case.

(C) Since $\begin{bmatrix} a_1 & 0 \\ 0 & d_1 \end{bmatrix}\begin{bmatrix} a_2 & 0 \\ 0 & d_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 & 0 \\ 0 & d_1d_2 \end{bmatrix}$, the statement is true.

(D) Since $\begin{bmatrix} a_1 & 0 \\ 0 & d_1 \end{bmatrix}\begin{bmatrix} a_2 & 0 \\ 0 & d_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 & 0 \\ 0 & d_1d_2 \end{bmatrix} = \begin{bmatrix} a_2a_1 & 0 \\ 0 & d_2d_1 \end{bmatrix} = \begin{bmatrix} a_2 & 0 \\ 0 & d_2 \end{bmatrix}\begin{bmatrix} a_1 & 0 \\ 0 & d_1 \end{bmatrix}$,

the statement is true.

73. $\frac{1}{2}(A+B) = \frac{1}{2}\left(\begin{bmatrix} 30 & 25 \\ 60 & 80 \end{bmatrix} + \begin{bmatrix} 36 & 27 \\ 54 & 74 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 66 & 52 \\ 114 & 154 \end{bmatrix} = \begin{bmatrix} 33 & 26 \\ 57 & 77 \end{bmatrix}$

This result provides the average cost of production for the two plants.

75. If a quantity is increased by 15%, the result is a multiplication by 1.15.

If a quantity is increased by 10%, the result is a multiplication by 1.1.

Thus we must calculate $1.1N - 1.15M$. The mark–up matrix is:

$$1.1N - 1.15M = 1.1\begin{bmatrix} 13{,}900 & 783 & 263 & 215 \\ 15{,}000 & 838 & 395 & 236 \\ 18{,}300 & 967 & 573 & 248 \end{bmatrix} - 1.15\begin{bmatrix} 10{,}400 & 682 & 215 & 182 \\ 12{,}500 & 721 & 295 & 182 \\ 16{,}400 & 827 & 443 & 192 \end{bmatrix}$$

$$= \begin{bmatrix} 15{,}290 & 861.3 & 289.3 & 236.5 \\ 16{,}500 & 921.8 & 434.5 & 259.6 \\ 20{,}130 & 1{,}063.7 & 630.3 & 272.8 \end{bmatrix} - \begin{bmatrix} 11{,}960 & 784.3 & 247.25 & 209.3 \\ 14{,}375 & 829.15 & 339.25 & 209.3 \\ 18{,}860 & 951.05 & 509.45 & 220.8 \end{bmatrix}$$

	Basic Car	Air	CD changer	Cruise Control
Model A	\$3,330	\$77	\$42	\$27
Model B	\$2,125	\$93	\$95	\$50
Model C	\$1,270	\$113	\$121	\$52

= Mark up

77. (A) $[0.6\ \ 0.6\ \ 0.2]\begin{bmatrix} 8 \\ 10 \\ 5 \end{bmatrix} = (0.6)8 + (0.6)10 + (0.2)5 = 11.80$ dollars per boat

(B) $[1.5\ \ 1.2\ \ 0.4]\begin{bmatrix} 9 \\ 12 \\ 6 \end{bmatrix} = (1.5)9 + (1.2)12 + (0.4)6 = 30.30$ dollars per boat

(C) The matrix NM has no obvious meaning, but the matrix MN gives the labor costs per boat at each plant.

(D) $MN = \begin{bmatrix} 0.6 & 0.6 & 0.2 \\ 1.0 & 0.9 & 0.3 \\ 1.5 & 1.2 & 0.4 \end{bmatrix}\begin{bmatrix} 8 & 9 \\ 10 & 12 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} (0.6)8+(0.6)10+(0.2)5 & (0.6)9+(0.6)12+(0.2)6 \\ (1.0)8+(0.9)10+(0.3)5 & (1.0)9+(0.9)12+(0.3)6 \\ (1.5)8+(1.2)10+(0.4)5 & (1.5)9+(1.2)12+(0.4)6 \end{bmatrix}$

$$= \begin{bmatrix} \$11.80 & \$13.80 \\ \$18.50 & \$21.60 \\ \$26.00 & \$30.30 \end{bmatrix} \begin{matrix} \text{One-person boat} \\ \text{Two-person boat} \\ \text{Four-person boat} \end{matrix}$$

(columns: Plant I, Plant II)

This matrix gives the labor costs for each type of boat at each plant.

79. (A) $A^2 = AA = \begin{bmatrix} 0&1&0&1&0 \\ 0&0&1&0&0 \\ 1&0&0&0&1 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \end{bmatrix} \begin{bmatrix} 0&1&0&1&0 \\ 0&0&1&0&0 \\ 1&0&0&0&1 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \end{bmatrix} = \begin{bmatrix} 0&0&2&0&0 \\ 1&0&0&0&1 \\ 0&1&0&2&0 \\ 1&0&0&0&1 \\ 0&0&1&0&0 \end{bmatrix}$

The 1 in row 2 and column 1 of A^2 indicates that there is one way to travel from Baltimore to Atlanta with one intermediate connection. The 2 in row 1 and column 3 indicates that there are two ways to travel from Atlanta to Chicago with one intermediate connection. In general, the elements in A^2 indicate the number of different ways to travel from the ith city to the jth city with one intermediate connection.

(B) $A^3 = A^2A = \begin{bmatrix} 0&0&2&0&0 \\ 1&0&0&0&1 \\ 0&1&0&2&0 \\ 1&0&0&0&1 \\ 0&0&1&0&0 \end{bmatrix} \begin{bmatrix} 0&1&0&1&0 \\ 0&0&1&0&0 \\ 1&0&0&0&1 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \end{bmatrix} = \begin{bmatrix} 2&0&0&0&2 \\ 0&1&0&2&0 \\ 0&0&3&0&0 \\ 0&1&0&2&0 \\ 1&0&0&0&1 \end{bmatrix}$

The 1 in row 4 and column 2 of A^3 indicates that there is one way to travel from Denver to Baltimore with two intermediate connections. The 2 in row 1 and column 5 indicates that there are two ways to travel from Atlanta to El Paso with two intermediate connections. In general, the elements in A^3 indicate the number of different ways to travel from the ith city to the the jth city with two intermediate connections.

(C) A is given above.

$$A + A^2 = \begin{bmatrix} 0&1&0&1&0 \\ 0&0&1&0&0 \\ 1&0&0&0&1 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \end{bmatrix} + \begin{bmatrix} 0&0&2&0&0 \\ 1&0&0&0&1 \\ 0&1&0&2&0 \\ 1&0&0&0&1 \\ 0&0&1&0&0 \end{bmatrix} = \begin{bmatrix} 0&1&2&1&0 \\ 1&0&1&0&1 \\ 1&1&0&2&1 \\ 1&0&1&0&1 \\ 0&0&1&1&0 \end{bmatrix}$$

$$A + A^2 + A^3 = \begin{bmatrix} 0&1&2&1&0 \\ 1&0&1&0&1 \\ 1&1&0&2&1 \\ 1&0&1&0&1 \\ 0&0&1&1&0 \end{bmatrix} + \begin{bmatrix} 2&0&0&0&2 \\ 0&1&0&2&0 \\ 0&0&3&0&0 \\ 0&1&0&2&0 \\ 1&0&0&0&1 \end{bmatrix} = \begin{bmatrix} 2&1&2&1&2 \\ 1&1&1&2&1 \\ 1&1&3&2&1 \\ 1&1&1&2&1 \\ 1&0&1&1&1 \end{bmatrix}$$

A zero element remains, so we must compute A^4.

$$A^4 = A^3A = \begin{bmatrix} 2&0&0&0&2 \\ 0&1&0&2&0 \\ 0&0&3&0&0 \\ 0&1&0&2&0 \\ 1&0&0&0&1 \end{bmatrix} \begin{bmatrix} 0&1&0&1&0 \\ 0&0&1&0&0 \\ 1&0&0&0&1 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \end{bmatrix} = \begin{bmatrix} 0&2&0&4&0 \\ 0&0&3&0&0 \\ 3&0&0&0&3 \\ 0&0&3&0&0 \\ 0&1&0&2&0 \end{bmatrix}$$

Then $A + A^2 + A^3 + A^4 = \begin{bmatrix} 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 3 & 2 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 0 & 4 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 3 & 0 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 & 5 & 2 \\ 1 & 1 & 4 & 2 & 1 \\ 4 & 1 & 3 & 2 & 4 \\ 1 & 1 & 4 & 2 & 1 \\ 1 & 1 & 1 & 3 & 1 \end{bmatrix}$

This matrix indicates that it is possible to travel from any origin to any destination with at most 3 intermediate connections.

81. (A) $[1{,}000 \;\; 500 \;\; 5{,}000]\begin{bmatrix} \$0.80 \\ \$1.50 \\ \$0.40 \end{bmatrix} = 1{,}000(\$0.80) + 500(\$1.50) + 5{,}000(\$0.40) = \$3{,}550$

(B) $[2{,}000 \;\; 800 \;\; 8{,}000]\begin{bmatrix} \$0.80 \\ \$1.50 \\ \$0.40 \end{bmatrix} = 2{,}000(\$0.80) + 800(\$1.50) + 8{,}000(\$0.40) = \$6{,}000$

(C) The matrix *MN* has no obvious interpretations, but the matrix *NM* represents the total cost of all contacts in each town.

(D) $NM = \begin{bmatrix} 1{,}000 & 500 & 5{,}000 \\ 2{,}000 & 800 & 8{,}000 \end{bmatrix}\begin{bmatrix} \$0.80 \\ \$1.50 \\ \$0.40 \end{bmatrix} = \begin{bmatrix} 1{,}000(0.80) + 500(1.50) + 5{,}000(0.40) \\ 2{,}000(0.80) + 800(1.50) + 8{,}000(0.40) \end{bmatrix}$

$= \begin{bmatrix} \$3{,}550 \\ \$6{,}000 \end{bmatrix} \begin{matrix} \text{Berkeley} \\ \text{Oakland} \end{matrix}$ = cost of all contacts in each town.

(E) The matrix $[1 \;\; 1]N$ can be used to find the total number of each of the three types of contact:

$[1 \;\; 1]\begin{bmatrix} 1{,}000 & 500 & 5{,}000 \\ 2{,}000 & 800 & 8{,}000 \end{bmatrix} = [1{,}000 + 2{,}000 \;\; 500 + 800 \;\; 5{,}000 + 8{,}000]$

[Telephone House Letter] = [3,000 1,300 13,000]

(F) The matrix $N\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ can be used to find the total number of contacts in each town:

$\begin{bmatrix} 1{,}000 & 500 & 5{,}000 \\ 2{,}000 & 800 & 8{,}000 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1{,}000 + 500 + 5{,}000 \\ 2{,}000 + 800 + 8{,}000 \end{bmatrix} = \begin{bmatrix} 6{,}500 \\ 10{,}800 \end{bmatrix} = \begin{bmatrix} \text{Berkeley contacts} \\ \text{Oakland contacts} \end{bmatrix}$

83. (A) Since player 1 did not defeat player 1, a 0 is placed in row 1, column 1.
Since player 1 did not defeat player 2, a 0 is placed in row 1, column 2.
Since player 1 defeated player 3, a 1 is placed in row 1, column 3.
Since player 1 defeated player 4, a 1 is placed in row 1, column 4.
Since player 1 defeated player 5, a 1 is placed in row 1, column 5.
Since player 1 did not defeat player 6, a 0 is placed in row 1, column 6.
Proceeding in this manner, we obtain

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{(B) } A^2 = \begin{bmatrix} 0&0&1&1&1&0\\ 1&0&0&1&1&0\\ 0&1&0&1&0&0\\ 0&0&0&0&0&1\\ 0&0&1&1&0&1\\ 1&1&1&0&0&0 \end{bmatrix}\begin{bmatrix} 0&0&1&1&1&0\\ 1&0&0&1&1&0\\ 0&1&0&1&0&0\\ 0&0&0&0&0&1\\ 0&0&1&1&0&1\\ 1&1&1&0&0&0 \end{bmatrix} = \begin{bmatrix} 0&1&1&2&0&2\\ 0&0&2&2&1&2\\ 1&0&0&1&1&1\\ 1&1&1&0&0&0\\ 1&2&1&1&0&1\\ 1&1&1&3&2&0 \end{bmatrix}$$

$$A + A^2 = \begin{bmatrix} 0&1&2&3&1&2\\ 1&0&2&3&2&2\\ 1&1&0&2&1&1\\ 1&1&1&0&0&1\\ 1&2&2&2&0&2\\ 2&2&2&3&2&0 \end{bmatrix} = B$$

Check (using graphing calculator):

```
[A]
 [[0 0 1 1 1 0]
  [1 0 0 1 1 0]
  [0 1 0 1 0 0]
  [0 0 0 0 0 1]
  [0 0 1 1 0 1]
  [1 1 1 0 0 0]]
```

```
[A]²
 [[0 1 1 2 0 2]
  [0 0 2 2 1 2]
  [1 0 0 1 1 1]
  [1 1 1 0 0 0]
  [1 2 1 1 0 1]
  [1 1 1 3 2 0]]
```

```
[A]+[A]²→[B]
 [[0 1 2 3 1 2]
  [1 0 2 3 2 2]
  [1 1 0 2 1 1]
  [1 1 1 0 0 1]
  [1 2 2 2 0 2]
  [2 2 2 3 2 0]]
```

Finally $B^{-1}A^{-1}$ is calculated as $C = \begin{bmatrix} 1\\1\\1\\1\\1\\1 \end{bmatrix}$ Then $BC = \begin{bmatrix} 0&1&2&3&1&2\\ 1&0&2&3&2&2\\ 1&1&0&2&1&1\\ 1&1&1&0&0&1\\ 1&2&2&2&0&2\\ 2&2&2&3&2&0 \end{bmatrix}\begin{bmatrix} 1\\1\\1\\1\\1\\1 \end{bmatrix} = \begin{bmatrix} 9\\10\\6\\4\\9\\11 \end{bmatrix}$

(D) *BC* measures the relative strength of the players, with the larger numbers representing greater strength. Thus, player 6 is the strongest and player 4 the weakest; ranking: Frank, Bart, Aaron & Elvis (tie), Charles, Dan.

Section 10–4

1. An identity matrix is a square matrix *I* whose product with any matrix *A*, if defined, is *A*.

3. The inverse matrix A^{-1} of a matrix A is a matrix such that $AA^{-1} = A^{-1}A = I$. Not every matrix has an inverse.

5. Answers will vary. **7.** Answers will vary. **9.** Gauss–Jordan elimination is the best approach.

11. $\begin{bmatrix} 2 & -3\\ 4 & 5 \end{bmatrix}$ **13.** $\begin{bmatrix} -2 & 1 & 3\\ 2 & 4 & -2\\ 5 & 1 & 0 \end{bmatrix}$ **15.** $\begin{bmatrix} 3 & -4\\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4\\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3\cdot 3+(-4)2 & 3\cdot 4+(-4)3\\ (-2)3+3\cdot 2 & (-2)4+3\cdot 3 \end{bmatrix} = \begin{bmatrix} 1 & 0\\ 0 & 1 \end{bmatrix}$

Thus, these two matrices are inverses of each other.

17. $\begin{bmatrix} 2 & 2\\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1\\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2\cdot 1+2(-1) & 2\cdot 1+2(-1)\\ (-1)1+(-1)(-1) & (-1)1+(-1)(-1) \end{bmatrix} = \begin{bmatrix} 0 & 0\\ 0 & 0 \end{bmatrix}$

Thus, these two matrices are not inverses of each other.

19. $\begin{bmatrix} -5 & 2\\ -8 & 3 \end{bmatrix}\begin{bmatrix} 3 & -2\\ 8 & -5 \end{bmatrix} = \begin{bmatrix} (-5)3+2\cdot 8 & (-5)(-2)+2(-5)\\ (-8)3+3\cdot 8 & (-8)(-2)+3(-5) \end{bmatrix} = \begin{bmatrix} 1 & 0\\ 0 & 1 \end{bmatrix}$

Thus, these two matrices are inverses of each other.

21. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1\cdot1+2\cdot0+0\cdot1 & 1(-2)+2\cdot1+0(-1) & 1\cdot0+2\cdot0+0\cdot0 \\ 0\cdot1+1\cdot0+0\cdot1 & 0(-2)+1\cdot1+0(-1) & 0\cdot0+1\cdot0+0\cdot0 \\ (-1)1+(-1)0+1\cdot1 & (-1)(-2)+(-1)1+1(-1) & (-1)0+(-1)0+1\cdot0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Thus, these two matrices are not inverses of each other.

23. $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} = \begin{bmatrix} 1\cdot3+(-1)(-2)+1(-4) & 1\cdot3+(-1)(-2)+1(-5) & 1(-1)+(-1)1+1\cdot2 \\ 0\cdot3+2(-2)+(-1)(-4) & 0\cdot3+2(-2)+(-1)(-5) & 0(-1)+2\cdot1+(-1)2 \\ 2\cdot3+3(-2)+0(-4) & 2\cdot3+3(-2)+0(-5) & 2(-1)+3\cdot1+0\cdot2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Thus, these two matrices are inverses of each other.

25. $2x_1 - x_2 = 3$
$x_1 + 3x_2 = -2$

27. $-2x_1 + x_3 = 3$
$x_1 + 2x_2 + x_3 = -4$
$x_2 - x_3 = 2$

29. $\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

31. $\begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$

33. Since $\begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3(-2) & (-2)1 \\ 1(-2) & 4\cdot1 \end{bmatrix} = \begin{bmatrix} -8 \\ 2 \end{bmatrix}$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -8 \\ 2 \end{bmatrix}$ if and only if $x_1 = -8$ and $x_2 = 2$.

35. Since $\begin{bmatrix} -2 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} (-2)3 & 3\cdot2 \\ 2\cdot3 & (-1)2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$ if and only if $x_1 = 0$ and $x_2 = 4$.

37. $\underset{A}{\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}}\underset{X}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{B}{\begin{bmatrix} 5 \\ 7 \end{bmatrix}}$

$AX = B$ has solution $X = A^{-1}B$.

To find A^{-1}, we perform row operations on

$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{array}\right] (-1)R_1 + R_2 \to R_2$

$\sim \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right] (-1)R_2 + R_1 \to R_1$

$\sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 1 \end{array}\right] (-1)R_2 \to R_2$

$\sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & -1 \end{array}\right]$

Hence $A^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$

Check: $A^{-1}A = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Then

$\underset{X}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{A^{-1}}{\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}}\underset{B}{\begin{bmatrix} 5 \\ 7 \end{bmatrix}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ $x_1 = 3, x_2 = -2.$

39. $\underset{A}{\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}}\underset{X}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{B}{\begin{bmatrix} 15 \\ 10 \end{bmatrix}}$

$AX = B$ has solution $X = A^{-1}B$.

To find A^{-1}, we perform row operations on

$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{array}\right] (-2)R_1 + R_2 \to R_2$

$\sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -5 & -2 & 1 \end{array}\right] \left(-\frac{1}{5}\right)R_2 \to R_2$

$\sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{2}{5} & -\frac{1}{5} \end{array}\right] (-1)R_2 + R_1 \to R_1$

$\sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{5} & \frac{1}{5} \\ 0 & 1 & \frac{2}{5} & -\frac{1}{5} \end{array}\right]$

Hence $A^{-1} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$

Check: $A^{-1}A = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Then

$\underset{X}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{A^{-1}}{\begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}}\underset{B}{\begin{bmatrix} 15 \\ 10 \end{bmatrix}} = \begin{bmatrix} 11 \\ 4 \end{bmatrix}$ $x_1 = 11, x_2 = 4$

41. $\left[\begin{array}{cc|cc} 1 & 9 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] (-9)R_2 + R_1 \to R_1 \sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & -9 \\ 0 & 1 & 0 & 1 \end{array}\right]$ Hence, $M^{-1} = \begin{bmatrix} 1 & -9 \\ 0 & 1 \end{bmatrix}$

Check: $M^{-1}M = \begin{bmatrix} 1 & -9 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 9 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1+(-9)0 & 1\cdot 9+(-9)1 \\ 0\cdot 1+1\cdot 0 & 0\cdot 9+1\cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

43. $\left[\begin{array}{cc|cc} -1 & -2 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right] 2R_1 + R_2 \to R_2 \sim \left[\begin{array}{cc|cc} -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right] 2R_2 + R_1 \to R_1 \sim \left[\begin{array}{cc|cc} -1 & 0 & 5 & 2 \\ 0 & 1 & 2 & 1 \end{array}\right] (-1)R_1 \to R_1$

$\sim \left[\begin{array}{cc|cc} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \end{array}\right]$ Hence, $M^{-1} = \begin{bmatrix} -5 & -2 \\ 2 & 1 \end{bmatrix}$

Check: $M^{-1}M = \begin{bmatrix} -5 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} (-5)(-1)+(-2)2 & (-5)(-2)+(-2)5 \\ 2(-1)+1\cdot 2 & 2(-2)+1\cdot 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

45. $\left[\begin{array}{cc|cc} -5 & 7 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{array}\right] 3R_2 + R_1 \to R_1 \sim \left[\begin{array}{cc|cc} 1 & -2 & 1 & 3 \\ 2 & -3 & 0 & 1 \end{array}\right] (-2)R_1 + R_2 \to R_2 \sim \left[\begin{array}{cc|cc} 1 & -2 & 1 & 3 \\ 0 & 1 & -2 & -5 \end{array}\right] 2R_2 + R_1 \to R_1$

$\sim \left[\begin{array}{cc|cc} 1 & 0 & -3 & -7 \\ 0 & 1 & -2 & -5 \end{array}\right]$ Hence, $M^{-1} = \begin{bmatrix} -3 & -7 \\ -2 & -5 \end{bmatrix}$

Check: $M^{-1}M = \begin{bmatrix} -3 & -7 \\ -2 & -5 \end{bmatrix}\begin{bmatrix} -5 & 7 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} (-3)(-5)+(-7)2 & (-3)7+(-7)(-3) \\ (-2)(-5)+(-5)2 & (-2)7+(-5)(-3) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

47. If the inverse existed we would find it by row operations on the following matrix: $= \left[\begin{array}{cc|cc} 3 & 9 & 1 & 0 \\ 2 & 6 & 0 & 1 \end{array}\right]$

But consider what happens if we perform $(-\frac{2}{3})R_1 + R_2 \to R_2 = \left[\begin{array}{cc|cc} 3 & 9 & 1 & 0 \\ 0 & 0 & -\frac{2}{3} & 1 \end{array}\right]$

Since a row of zeros results to the left of the vertical line, no inverse exists.

49. $\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right] \frac{1}{2}R_1 \to R_1 \sim \left[\begin{array}{cc|cc} 1 & 1.5 & 0.5 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right] (-3)R_1 + R_2 \to R_2 \sim \left[\begin{array}{cc|cc} 1 & 1.5 & 0.5 & 0 \\ 0 & 0.5 & -1.5 & 1 \end{array}\right] 2R_2 \to R_2$

$\sim \left[\begin{array}{cc|cc} 1 & 1.5 & 0.5 & 0 \\ 0 & 1 & -3 & 2 \end{array}\right] (-1.5)R_2 + R_1 \to R_1 \sim \left[\begin{array}{cc|cc} 1 & 0 & 5 & -3 \\ 0 & 1 & -3 & 2 \end{array}\right]$ The inverse is $\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$

The checking steps are omitted for lack of space in this and some subsequent problems.

51. $\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] R_1 + R_2 \to R_2 \sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] R_2 \leftrightarrow R_3$

$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 1 & 0 \end{array}\right] (-1)R_2 + R_1 \to R_1 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} (-1)R_3 + R_1 \to R_1 \\ R_3 + R_2 \to R_2 \end{array}$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} (-1)R_2 \to R_2 \\ (-1)R_3 \to R_3 \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{array}\right]$ Hence, $M^{-1} = \begin{bmatrix} 0 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 0 \end{bmatrix}$

Check: $M^{-1}M = \begin{bmatrix} 0 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 0\cdot 1+(-1)(-1)+(-1)0 & 0(-1)+(-1)1+(-1)(-1) & 0\cdot 0+(-1)(-1)+(-1)1 \\ (-1)1+(-1)(-1)+(-1)0 & (-1)(-1)+(-1)1+(-1)(-1) & (-1)0+(-1)(-1)+(-1)1 \\ (-1)1+(-1)(-1)+0\cdot 0 & (-1)(-1)+(-1)1+0(-1) & (-1)0+(-1)(-1)+0\cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

53. $\left[\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 3 & 5 & 9 & 0 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ (-3)R_1 + R_2 \to R_2 \\ (-1)R_1 + R_3 \to R_3 \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 0 & -1 & -6 & -3 & 1 & 0 \\ 0 & -1 & -7 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} 2R_2 + R_1 \to R_1 \\ \\ (-1)R_2 + R_3 \to R_3 \end{array}$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -7 & -5 & 2 & 0 \\ 0 & -1 & -6 & -3 & 1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 1 \end{array}\right] \begin{array}{l} (-7)R_3 + R_1 \to R_1 \\ (-6)R_3 + R_2 \to R_2 \\ \\ \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -19 & 9 & -7 \\ 0 & -1 & 0 & -15 & 7 & -6 \\ 0 & 0 & -1 & 2 & -1 & 1 \end{array}\right] \begin{array}{l} \\ (-1)R_2 \to R_2 \\ (-1)R_3 \to R_3 \end{array}$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -19 & 9 & -7 \\ 0 & 1 & 0 & 15 & -7 & 6 \\ 0 & 0 & 1 & -2 & 1 & -1 \end{array}\right]$ Hence, $M^{-1} = \begin{bmatrix} -19 & 9 & -7 \\ 15 & -7 & 6 \\ -2 & 1 & -1 \end{bmatrix}$

Check: $M^{-1}M = \begin{bmatrix} -19 & 9 & -7 \\ 15 & -7 & 6 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ 3 & 5 & 9 \\ 1 & 1 & -2 \end{bmatrix}$

$$= \begin{bmatrix} (-19)1 + 9\cdot 3 + (-7)1 & (-19)2 + 9\cdot 5 + (-7)1 & (-19)5 + 9\cdot 9 + (-7)(-2) \\ 15\cdot 1 + (-7)3 + 6\cdot 1 & 15\cdot 2 + (-7)5 + 6\cdot 1 & 15\cdot 5 + (-7)9 + 6(-2) \\ (-2)1 + 1\cdot 3 + (-1)1 & (-2)2 + 1\cdot 5 + (-1)1 & (-2)5 + 1\cdot 9 + (-1)(-2) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

55. $\left[\begin{array}{ccc|ccc} 2 & 2 & -1 & 1 & 0 & 0 \\ 0 & 4 & -1 & 0 & 1 & 0 \\ -1 & -2 & 1 & 0 & 0 & 1 \end{array}\right] R_1 \leftrightarrow R_3 \sim \left[\begin{array}{ccc|ccc} -1 & -2 & 1 & 0 & 0 & 1 \\ 0 & 4 & -1 & 0 & 1 & 0 \\ 2 & 2 & -1 & 1 & 0 & 0 \end{array}\right] 2R_1 + R_3 \to R_3$

$\sim \left[\begin{array}{ccc|ccc} -1 & -2 & 1 & 0 & 0 & 1 \\ 0 & 4 & -1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 1 & 0 & 2 \end{array}\right] \begin{array}{l} (-1)R_3 + R_1 \to R_1 \\ 2R_3 + R_2 \to R_2 \\ \\ \end{array} \sim \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 2 & 1 & 4 \\ 0 & -2 & 1 & 1 & 0 & 2 \end{array}\right] (-1)R_2 + R_3 \to R_3$

$\sim \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 2 & 1 & 4 \\ 0 & -2 & 0 & -1 & -1 & -2 \end{array}\right] R_2 \leftrightarrow R_3 \sim \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & -2 & 0 & -1 & -1 & -2 \\ 0 & 0 & 1 & 2 & 1 & 4 \end{array}\right] \begin{array}{l} (-1)R_1 \to R_1 \\ (-\frac{1}{2})R_2 \to R_2 \\ \\ \end{array}$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 0 & 1 & 2 & 1 & 4 \end{array}\right]$ The inverse is $\begin{bmatrix} 1 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ 2 & 1 & 4 \end{bmatrix}$

57. If the inverse existed we would find it by row operations on the following matrix: $\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$

But consider what happens if we perform $R_2 + R_3 \to R_2$ $\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$

Since a row of zeros results to the left of the vertical line, no inverse exists.

59. $\left[\begin{array}{ccc|ccc} 1 & 5 & 10 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 1 & 6 & 15 & 0 & 0 & 1 \end{array}\right] (-1)R_1 + R_3 \to R_3 \sim \left[\begin{array}{ccc|ccc} 1 & 5 & 10 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 1 & 5 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} (-5)R_2 + R_1 \to R_1 \\ \\ (-1)R_2 + R_3 \to R_3 \end{array}$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -10 & 1 & -5 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \begin{array}{l} 10R_3 + R_1 \to R_1 \\ (-4)R_3 + R_2 \to R_2 \\ \\ \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9 & -15 & 10 \\ 0 & 1 & 0 & 4 & 5 & -4 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$ The inverse is $\begin{bmatrix} -9 & -15 & 10 \\ 4 & 5 & -4 \\ -1 & -1 & 1 \end{bmatrix}$

61. $\begin{bmatrix} -1 & -2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$

$AX = K$ has solution $X = A^{-1}K$.

We find $A^{-1}K$ for each given K. From problem 43, $A^{-1} = \begin{bmatrix} -5 & -2 \\ 2 & 1 \end{bmatrix}$

(A) $K = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ $A^{-1}K = \begin{bmatrix} -5 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -20 \\ 9 \end{bmatrix}$ $x_1 = -20, x_2 = 9$

(B) $K = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ $A^{-1}K = \begin{bmatrix} -5 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 18 \\ -7 \end{bmatrix}$ $x_1 = 18, x_2 = -7$

(C) $K = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$ $A^{-1}K = \begin{bmatrix} -5 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 19 \\ -8 \end{bmatrix}$ $x_1 = 19, x_2 = -8$

63. $\begin{bmatrix} -5 & 7 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$

$AX = K$ has solution $X = A^{-1}K$.

We find $A^{-1}K$ for each given K. From problem 45, $A^{-1} = \begin{bmatrix} -3 & -7 \\ -2 & -5 \end{bmatrix}$

(A) $K = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$ $A^{-1}K = \begin{bmatrix} -3 & -7 \\ -2 & -5 \end{bmatrix}\begin{bmatrix} -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$ $x_1 = 8, x_2 = 5$

(B) $K = \begin{bmatrix} 8 \\ -4 \end{bmatrix}$ $A^{-1}K = \begin{bmatrix} -3 & -7 \\ -2 & -5 \end{bmatrix}\begin{bmatrix} 8 \\ -4 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ $x_1 = 4, x_2 = 4$

(C) $K = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$ $A^{-1}K = \begin{bmatrix} -3 & -7 \\ -2 & -5 \end{bmatrix}\begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} -18 \\ -12 \end{bmatrix}$ $x_1 = -18, x_2 = -12$

65. $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$

$AX = K$ has solution $X = A^{-1}K$.

We find $A^{-1}K$ for each given K. From problem 51, $A^{-1} = \begin{bmatrix} 0 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 0 \end{bmatrix}$

(A) $K = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ $A^{-1}K = \begin{bmatrix} 0 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \\ -2 \end{bmatrix}$ $x_1 = -3, x_2 = -4, x_3 = -2$

(B) $K = \begin{bmatrix} -1 \\ 0 \\ -4 \end{bmatrix}$ $A^{-1}K = \begin{bmatrix} 0 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}$ $x_1 = 4, x_2 = 5, x_3 = 1$

(C) $K = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$ $A^{-1}K = \begin{bmatrix} 0 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$ $x_1 = 2, x_2 = -1, x_3 = -1$

67. $\begin{bmatrix} 1 & 2 & 5 \\ 3 & 5 & 9 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$

$AX = K$ has solution $X = A^{-1}K$.

We find $A^{-1}K$ for each given K. From problem 53, $A^{-1} = \begin{bmatrix} -19 & 9 & -7 \\ 15 & -7 & 6 \\ -2 & 1 & -1 \end{bmatrix}$

(A) $K = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$ $A^{-1}K = \begin{bmatrix} -19 & 9 & -7 \\ 15 & -7 & 6 \\ -2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -19 \\ 17 \\ -3 \end{bmatrix}$ $x_1 = -19, x_2 = 17, x_3 = -3$

(B) $K = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$ $A^{-1}K = \begin{bmatrix} -19 & 9 & -7 \\ 15 & -7 & 6 \\ -2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -104 \\ 82 \\ -11 \end{bmatrix}$ $x_1 = -104, x_2 = 82, x_3 = -11$

(C) $K = \begin{bmatrix} -6 \\ 0 \\ 2 \end{bmatrix}$ $A^{-1}K = \begin{bmatrix} -19 & 9 & -7 \\ 15 & -7 & 6 \\ -2 & 1 & -1 \end{bmatrix}\begin{bmatrix} -6 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 100 \\ -78 \\ 10 \end{bmatrix}$ $x_1 = 100, x_2 = -78, x_3 = 10$

69.

$AX = BX + C$	
$AX + (-BX) = (-BX) + BX + C$	Addition property of equality
$AX - BX = 0 + C$	Additive inverse property; definition of subtraction
$AX - BX = C$	
$(A - B)X = C$	Right distributive property

[To be more careful, we should write
$AX - BX = AX + -BX$ by the definition of subtraction
$= [A + (-B)]X$ by the right distributive property
$= (A - B)X$ by the definition of subtraction
but the distributive properties are generally understood as applying to subtraction also.]

$(A - B)^{-1}[(A - B)X] = (A - B)^{-1}C$	Left multiplication property of equality
$[(A - B)^{-1}(A - B)]X = (A - B)^{-1}C$	Associative property
$IX = (A - B)^{-1}C$	Multiplicative inverse property
$X = (A - B)^{-1}C$	Multiplicative identity property

71.

$X = AX + C$	
$X + (-AX) = (-AX) + AX + C$	Addition property of equality
$X - AX = 0 + C$	Additive inverse property; definition of subtraction
$X - AX = C$	Additive identity property
$IX - AX = C$	Multiplicative identity property
$(I - A)X = C$	Right distributive property
$(I - A)^{-1}[(I - A)X] = (I - A)^{-1}C$	Left multiplication property of equality
$[(I - A)^{-1}(I - A)]X = (I - A)^{-1}C$	Associative property
$IX = (I - A)^{-1}C$	Multiplicative inverse property
$X = (I - A)^{-1}C$	Multiplicative identity property

73.

$AX + C = 3X$	
$AX + (-AX) + C = 3X + (-AX)$	Addition property of equality
$0 + C = 3X - AX$	Additive inverse property; definition of subtraction
$C = 3X - AX$	Additive identity property
$C = 3IX - AX$	Multiplicative identity property
$C = (3I - A)X$	Right distributive property
$(3I - A)^{-1} C = (3I - A)^{-1}[(3I - A)X]$	Left multiplication property of equality
$(3I - A)^{-1} C = [(3I - A)^{-1}(3I - A)]X$	Associative property
$(3I - A)^{-1} C = IX$	Multiplicative inverse property
$(3I - A)^{-1} C = X$	Multiplicative inverse property

75. Try to find A^{-1}:

$\left[\begin{array}{cc|cc} a & 0 & 1 & 0 \\ 0 & d & 0 & 1 \end{array}\right] \frac{1}{a}R_1 \to R_1 \sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a} & 0 \\ 0 & d & 0 & 1 \end{array}\right] \frac{1}{d}R_2 \to R_2 \sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a} & 0 \\ 0 & 1 & 0 & \frac{1}{d} \end{array}\right]$ The inverse is $A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{bmatrix}$

This will exist unless either a or d is zero, which would make $\frac{1}{a}$ or $\frac{1}{d}$ undefined. So A^{-1} exists exactly when both a and d are non–zero.

77. (A) A^{-1}: $\left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ -4 & -3 & 0 & 1 \end{array}\right] R_2 + R_1 \to R_1$

$\sim \left[\begin{array}{cc|cc} -1 & -1 & 1 & 1 \\ -4 & -3 & 0 & 1 \end{array}\right] (-1)R_1 \to R_1$

$\sim \left[\begin{array}{cc|cc} 1 & 1 & -1 & -1 \\ -4 & -3 & 0 & 1 \end{array}\right] 4R_1 + R_2 \to R_2$

$\sim \left[\begin{array}{cc|cc} 1 & 1 & -1 & -1 \\ 0 & 1 & -4 & -3 \end{array}\right] (-1)R_2 + R_1 \to R_1$

$\sim \left[\begin{array}{cc|cc} 1 & 0 & 3 & 2 \\ 0 & 1 & -4 & -3 \end{array}\right] \quad A^{-1} = \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}$

A^2: $\begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}$

$= \begin{bmatrix} 3\cdot 3 + 2(-4) & 3\cdot 2 + 2(-3) \\ (-4)3 + (-3)(-4) & (-4)2 + (-3)(-3) \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(B) A^{-1}: $\left[\begin{array}{cc|cc} -2 & -1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] R_2 + R_1 \to R_1$

$\sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 3 & 2 & 0 & 1 \end{array}\right] (-3)R_1 + R_2 \to R_2$

$\sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 0 & -1 & -3 & -2 \end{array}\right] R_2 + R_1 \to R_1$

$\sim \left[\begin{array}{cc|cc} 1 & 0 & -2 & -1 \\ 0 & -1 & -3 & -2 \end{array}\right] (-1)R_2 \to R_2$

$\sim \left[\begin{array}{cc|cc} 1 & 0 & -2 & -1 \\ 0 & 1 & 3 & 2 \end{array}\right] \quad A^{-1} = \begin{bmatrix} -2 & -1 \\ 3 & 2 \end{bmatrix}$

A^2: $\begin{bmatrix} -2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -2 & -1 \\ 3 & 2 \end{bmatrix}$

$= \begin{bmatrix} (-2)(-2) + (-1)3 & (-2)(-1) + (-1)2 \\ 3(-2) + 2\cdot 3 & 3(-1) + 2\cdot 2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Note that in both cases $A^{-1} = A$ and $A^2 = I$.

79. (A) We calculate A^{-1} by row operations on

$$\left[\begin{array}{cc|cc}4&2&1&0\\1&3&0&1\end{array}\right] R_1 \leftrightarrow R_2$$

$$\sim \left[\begin{array}{cc|cc}1&3&0&1\\4&2&1&0\end{array}\right] (-4)R_1 + R_2 \to R_2$$

$$\sim \left[\begin{array}{cc|cc}1&3&0&1\\0&-10&1&-4\end{array}\right] \left(-\tfrac{1}{10}\right)R_2 \to R_2$$

$$\sim \left[\begin{array}{cc|cc}1&3&0&1\\0&1&-\frac{1}{10}&\frac{2}{5}\end{array}\right] (-3)R_2 + R_1 \to R_1$$

$$\sim \left[\begin{array}{cc|cc}1&0&\frac{3}{10}&-\frac{1}{5}\\0&1&-\frac{1}{10}&\frac{2}{5}\end{array}\right]$$

Hence $A^{-1} = \begin{bmatrix}\frac{3}{10}&-\frac{1}{5}\\-\frac{1}{10}&\frac{2}{5}\end{bmatrix}$

(B) We calculate A^{-1} by row operations on

$$\left[\begin{array}{cc|cc}5&5&1&0\\-1&3&0&1\end{array}\right] R_1 \leftrightarrow R_2$$

$$\sim \left[\begin{array}{cc|cc}-1&3&0&1\\5&5&1&0\end{array}\right] 5R_1 + R_2 \to R_2$$

$$\sim \left[\begin{array}{cc|cc}-1&3&0&1\\0&20&1&5\end{array}\right] \tfrac{1}{20}R_2 \to R_2$$

$$\sim \left[\begin{array}{cc|cc}-1&3&0&1\\0&1&\frac{1}{20}&\frac{1}{4}\end{array}\right] (-3)R_2 + R_1 \to R_1$$

$$\sim \left[\begin{array}{cc|cc}-1&0&-\frac{3}{20}&\frac{1}{4}\\0&1&\frac{1}{20}&\frac{1}{4}\end{array}\right] (-1)R_1 \to R_1$$

$$\sim \left[\begin{array}{cc|cc}1&0&\frac{3}{20}&-\frac{1}{4}\\0&1&\frac{1}{20}&\frac{1}{4}\end{array}\right]$$

Hence $A^{-1} = \begin{bmatrix}\frac{3}{20}&-\frac{1}{4}\\\frac{1}{20}&\frac{1}{4}\end{bmatrix}$

We calculate $(A^{-1})^{-1}$ by row operations on

$$\left[\begin{array}{cc|cc}\frac{3}{10}&-\frac{1}{5}&1&0\\-\frac{1}{10}&\frac{2}{5}&0&1\end{array}\right] \tfrac{10}{3}R_1 \to R_1$$

$$\sim \left[\begin{array}{cc|cc}1&-\frac{2}{3}&\frac{10}{3}&0\\-\frac{1}{10}&\frac{2}{5}&0&1\end{array}\right] \tfrac{1}{10}R_1 + R_2 \to R_2$$

$$\sim \left[\begin{array}{cc|cc}1&-\frac{2}{3}&\frac{10}{3}&0\\0&\frac{1}{3}&\frac{1}{3}&1\end{array}\right] 2R_2 + R_1 \to R_1$$

$$\sim \left[\begin{array}{cc|cc}1&0&4&2\\0&\frac{1}{3}&\frac{1}{3}&1\end{array}\right] 3R_2 \to R_2$$

$$\sim \left[\begin{array}{cc|cc}1&0&4&2\\0&1&1&3\end{array}\right]$$

Hence $(A^{-1})^{-1} = \begin{bmatrix}4&2\\1&3\end{bmatrix}$

We calculate A^{-1} by row operations on

$$\left[\begin{array}{cc|cc}\frac{3}{20}&-\frac{1}{4}&1&0\\\frac{1}{20}&\frac{1}{4}&0&1\end{array}\right] \tfrac{20}{3}R_1 \to R_1$$

$$\sim \left[\begin{array}{cc|cc}1&-\frac{5}{3}&\frac{20}{3}&0\\\frac{1}{20}&\frac{1}{4}&0&1\end{array}\right] -\tfrac{1}{20}R_1 + R_2 \to R_2$$

$$\sim \left[\begin{array}{cc|cc}1&-\frac{5}{3}&\frac{20}{3}&0\\0&\frac{1}{3}&-\frac{1}{3}&1\end{array}\right] 5R_2 + R_1 \to R_1$$

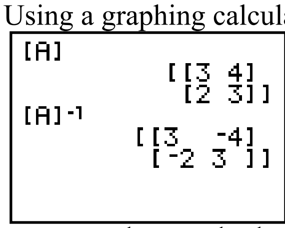
$$\sim \left[\begin{array}{cc|cc}1&0&5&5\\0&\frac{1}{3}&-\frac{1}{3}&1\end{array}\right] 3R_2 \to R_2$$

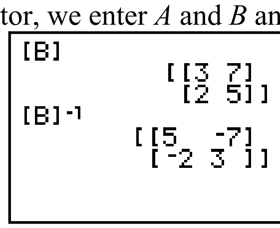
$$\sim \left[\begin{array}{cc|cc}1&0&5&5\\0&1&-1&3\end{array}\right]$$

Hence $(A^{-1})^{-1} = \begin{bmatrix}5&5\\-1&3\end{bmatrix}$

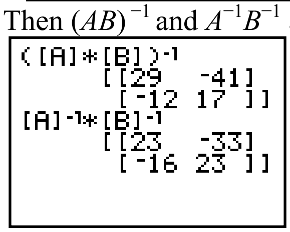
Note that in both cases the inverse of the inverse works out to be the original matrix.

81. (A) Using a graphing calculator, we enter A and B and calculate A^{-1} and B^{-1}

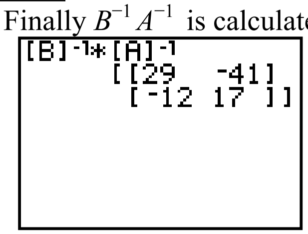




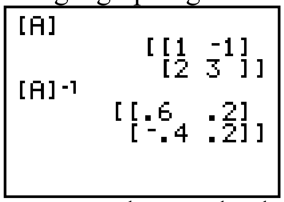
Then $(AB)^{-1}$ and $A^{-1}B^{-1}$ are calculated as

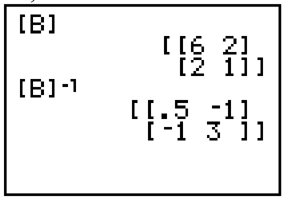


Finally $B^{-1}A^{-1}$ is calculated as

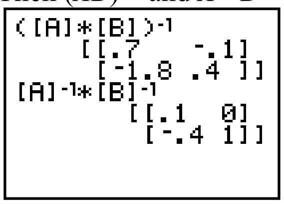


(B) Using a graphing calculator, we enter A and B and calculate A^{-1} and B^{-1}

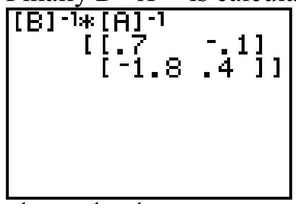




Then $(AB)^{-1}$ and $A^{-1}B^{-1}$ are calculated as



Finally $B^{-1}A^{-1}$ is calculated as



Notice that in each case $(AB)^{-1} = B^{-1}A^{-1}$ but $(AB)^{-1} \neq A^{-1}B^{-1}$.

83. Using the assignation numbers 1 to 27 with the letters of the alphabet and a blank as in the text, write
L E B R O N J A M E S
12 5 2 18 15 14 27 10 1 13 5 19
and calculate

$$\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 12 & 2 & 15 & 27 & 1 & 5 \\ 5 & 18 & 14 & 10 & 13 & 19 \end{bmatrix} = \begin{bmatrix} 61 & 96 & 115 & 131 & 68 & 110 \\ 22 & 38 & 43 & 47 & 27 & 43 \end{bmatrix}$$

The encoded message is thus
61 22 96 38 115 43 131 47 68 27 110 43

85. The inverse of matrix A is easily calculated to be $A^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$

Putting the coded message into matrix form and multiplying by A^{-1} yields:

$$\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 31 & 150 & 57 & 150 & 103 & 160 & 61 & 192 \\ 12 & 55 & 20 & 59 & 39 & 61 & 22 & 73 \end{bmatrix} = \begin{bmatrix} 2 & 25 & 14 & 5 & 11 & 15 & 12 & 19 \\ 5 & 15 & 3 & 27 & 14 & 23 & 5 & 27 \end{bmatrix}$$

This decodes to 2 5 25 15 14 3 5 27 11 14 15 23 12 5 19 27
B E Y O N C E K N O W L E S

87. Using the assignation of numbers 1 to 27 with the letters of the alphabet and a blank as in the text, write
N E W E N G L A N D P A T R I O T S
14 5 23 27 5 14 7 12 1 14 4 27 16 1 20 18 9 15 20 19
and calculate

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 3 \\ 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 1 & 1 & 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 14 & 14 & 4 & 18 \\ 5 & 7 & 27 & 9 \\ 23 & 12 & 16 & 15 \\ 27 & 1 & 1 & 20 \\ 5 & 14 & 20 & 19 \end{bmatrix} = \begin{bmatrix} 42 & 40 & 40 & 52 \\ 43 & 61 & 103 & 81 \\ 88 & 62 & 72 & 99 \\ 33 & 40 & 56 & 53 \\ 101 & 49 & 69 & 101 \end{bmatrix}$$

The coded message is thus
42 40 40 52 43 61 103 81 88 62 72 99 33 40 56 53 101 49 69 101.

89. The inverse of B is calculated to be

$$B^{-1} = \begin{bmatrix} -2 & -1 & 2 & 2 & -1 \\ 3 & 2 & -2 & -4 & 1 \\ 6 & 2 & -4 & -5 & 2 \\ -2 & -1 & 1 & 2 & 0 \\ -3 & -1 & 2 & 3 & -1 \end{bmatrix}$$

Putting the coded message into matrix form and multiplying by B^{-1} yields

$$\begin{bmatrix} -2 & -1 & 2 & 2 & -1 \\ 3 & 2 & -2 & -4 & 1 \\ 6 & 2 & -4 & -5 & 2 \\ -2 & -1 & 1 & 2 & 0 \\ -3 & -1 & 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 32 & 51 & 62 & 58 & 39 \\ 25 & 64 & 109 & 115 & 110 \\ 55 & 103 & 114 & 105 & 85 \\ 19 & 39 & 62 & 73 & 65 \\ 41 & 100 & 92 & 113 & 111 \end{bmatrix} = \begin{bmatrix} 18 & 18 & 27 & 12 & 1 \\ 1 & 19 & 20 & 15 & 18 \\ 9 & 27 & 8 & 19 & 11 \\ 4 & 15 & 5 & 20 & 27 \\ 5 & 6 & 27 & 27 & 27 \end{bmatrix}$$

This decodes to

18 1 9 4 5 18 19 27 15 6 27 20 8 5 27 12 15 19 20 27 1 18 11 27 27
R A I D E R S O F T H E L O S T A R K

91. The system to be solved, for an arbitrary return, is derived as follows:

Let x_1 = number of \$20 tickets sold
x_2 = number of \$30 tickets sold

Then $x_1 + x_2 = 10{,}000$ number of seats
$20x_1 + 30x_2 = k_2$ return required

We solve the system by writing it as a matrix equation.

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 20 & 30 \end{bmatrix}} \underset{X}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{B}{\begin{bmatrix} 10{,}000 \\ k_2 \end{bmatrix}}$$

If A^{-1} exists, then $X = A^{-1}B$. To find A^{-1}, we perform row operations on

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 20 & 30 & 0 & 1 \end{array}\right] (-20)R_1 + R_2 \to R_2 \sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 10 & -20 & 1 \end{array}\right] 0.1R_2 \to R_2 \sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0.1 \end{array}\right] (-1)R_2 + R_1 \to R_1$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 3 & -0.1 \\ 0 & 1 & -2 & 0.1 \end{array}\right]$$

Hence $A^{-1} = \begin{bmatrix} 3 & -0.1 \\ -2 & 0.1 \end{bmatrix}$

Check: $A^{-1}A = \begin{bmatrix} 3 & -0.1 \\ -2 & 0.1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 20 & 30 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

We can now solve the system as

$$\underset{X}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{A^{-1}}{\begin{bmatrix} 3 & -0.1 \\ -2 & 0.1 \end{bmatrix}} \underset{B}{\begin{bmatrix} 10{,}000 \\ k_2 \end{bmatrix}}$$

If $k_2 = 240{,}000$ (Concert 1),

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -0.1 \\ -2 & 0.1 \end{bmatrix} \begin{bmatrix} 10{,}000 \\ 240{,}000 \end{bmatrix} = \begin{bmatrix} 6{,}000 \\ 4{,}000 \end{bmatrix}$$

Concert 1: 6,000 \$20 tickets and 4,000 \$30 tickets

If $k_2 = 250{,}000$ (Concert 2),

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -0.1 \\ -2 & 0.1 \end{bmatrix} \begin{bmatrix} 10{,}000 \\ 250{,}000 \end{bmatrix} = \begin{bmatrix} 5{,}000 \\ 5{,}000 \end{bmatrix}$$

Concert 2: 5,000 \$20 tickets and 5,000 \$30 tickets

If $k_2 = 270{,}000$ (Concert 3),

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -0.1 \\ -2 & 0.1 \end{bmatrix} \begin{bmatrix} 10{,}000 \\ 270{,}000 \end{bmatrix} = \begin{bmatrix} 3{,}000 \\ 7{,}000 \end{bmatrix}$$

Concert 3: 3,000 \$20 tickets and 7,000 \$30 tickets

93. We solve the system, for arbitrary V_1 and V_2, by writing it as a matrix equation.

$$\overset{A}{\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}} \overset{J}{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}} = \overset{B}{\begin{bmatrix} 0 \\ V_1 \\ V_2 \end{bmatrix}}$$

If A^{-1} exists, then $J = A^{-1}B$. To find A^{-1}, we perform row operations on

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] (-1)R_1 + R_2 \to R_2 \sim \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] R_2 \leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 2 & -1 & -1 & 1 & 0 \end{array}\right] \begin{matrix} R_2 + R_1 \to R_1 \\ \\ (-2)R_2 + R_3 \to R_3 \end{matrix} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & -5 & -1 & 1 & -2 \end{array}\right] -\tfrac{1}{5}R_3 \to R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} & \frac{2}{5} \end{array}\right] \begin{matrix} (-3)R_3 + R_1 \to R_1 \\ (-2)R_3 + R_2 \to R_2 \end{matrix} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{5} & \frac{3}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & -\frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} & \frac{2}{5} \end{array}\right]$$

Hence $A^{-1} = \frac{1}{5}\begin{bmatrix} 2 & 3 & -1 \\ -2 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$ Check: $A^{-1}A = \frac{1}{5}\begin{bmatrix} 2 & 3 & -1 \\ -2 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We can now solve the system as

$$\overset{J}{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}} = \overset{A^{-1}}{\frac{1}{5}\begin{bmatrix} 2 & 3 & -1 \\ -2 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix}} \overset{B}{\begin{bmatrix} 0 \\ V_1 \\ V_2 \end{bmatrix}}$$

(A) $V_1 = 10 \quad V_2 = 10$

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & 3 & -1 \\ -2 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}$$

$I_1 = 4, I_2 = 6, I_3 = 2$ (amperes)

(B) $V_1 = 10 \quad V_2 = 15$

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & 3 & -1 \\ -2 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 10 \\ 15 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$$

$I_1 = 3, I_2 = 7, I_3 = 4$ (amperes)

(C) $V_1 = 15 \quad V_2 = 10$

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & 3 & -1 \\ -2 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 15 \\ 10 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 1 \end{bmatrix}$$

$I_1 = 7, I_2 = 8, I_3 = 1$ (amperes)

95. If the graph of $f(x) = ax^2 + bx + c$ passes through a point, the coordinates of the point must satisfy the equation of the graph. Hence

$k_1 = a(1)^2 + b(1) + c$
$k_2 = a(2)^2 + b(2) + c$
$k_3 = a(3)^2 + b(3) + c$

After simplification, we obtain:

$a + b + c = k_1$
$4a + 2b + c = k_2$
$9a + 3b + c = k_3$

We solve this system, for arbitrary k_1, k_2, k_3, by writing it as a matrix equation.

$$\overset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix}} \overset{X}{\begin{bmatrix} a \\ b \\ c \end{bmatrix}} = \overset{B}{\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}}$$

If A^{-1} exists, then $X = A^{-1}B$. To find A^{-1} we perform row operations on

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 \\ 9 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \\ (-4)R_1 + R_2 \to R_2 \\ (-9)R_1 + R_3 \to R_3 \end{matrix} \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -4 & 1 & 0 \\ 0 & -6 & -8 & -9 & 0 & 1 \end{array}\right] -\tfrac{1}{2}R_2 \to R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 2 & -\frac{1}{2} & 0 \\ 0 & -6 & -8 & -9 & 0 & 1 \end{array}\right] \begin{matrix} (-1)R_2 + R_1 \to R_1 \\ \\ 6R_2 + R_3 \to R_3 \end{matrix} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & -1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{2} & 2 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 3 & -3 & 1 \end{array}\right] \begin{matrix} \frac{1}{2}R_3 + R_1 \to R_1 \\ (-\frac{3}{2})R_3 + R_2 \to R_2 \\ \\ \end{matrix}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{5}{2} & 4 & -\frac{3}{2} \\ 0 & 0 & 1 & 3 & -3 & 1 \end{array}\right]$$

Hence $A^{-1} = \dfrac{1}{2}\begin{bmatrix} 1 & -2 & 1 \\ -5 & 8 & -3 \\ 6 & -6 & 2 \end{bmatrix}$

Check: $A^{-1}A = \dfrac{1}{2}\begin{bmatrix} 1 & -2 & 1 \\ -5 & 8 & -3 \\ 6 & -6 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We can now solve the system as

$$\overset{X}{\begin{bmatrix} a \\ b \\ c \end{bmatrix}} = \frac{1}{2}\overset{A^{-1}}{\begin{bmatrix} 1 & -2 & 1 \\ -5 & 8 & -3 \\ 6 & -6 & 2 \end{bmatrix}} \overset{B}{\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}}$$

(A) $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} 1 & -2 & 1 \\ -5 & 8 & -3 \\ 6 & -6 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$ $a = 1, b = 0, c = -3$

(B) $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} 1 & -2 & 1 \\ -5 & 8 & -3 \\ 6 & -6 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}$ $a = -2, b = 5, c = 1$

(C) $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} 1 & -2 & 1 \\ -5 & 8 & -3 \\ 6 & -6 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 11 \\ -46 \\ 43 \end{bmatrix}$ $a = 11, b = -46, c = 43$

97. The system to be solved, for an arbitrary diet, is derived as follows:
Let x_1 = amount of mix A
x_2 = amount of mix B
Then $0.20x_1 + 0.10x_2 = k_1$ (k_1 = amount of protein)
$0.02x_1 + 0.06x_2 = k_2$ (k_2 = amount of fat)
We solve the system by writing it as a matrix equation.

$$\underset{A}{\begin{bmatrix} 0.20 & 0.10 \\ 0.02 & 0.06 \end{bmatrix}} \underset{X}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{B}{\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}}$$

If A^{-1} exists, then $X = A^{-1}B$. To find A^{-1}, we perform row operations on

$$\left[\begin{array}{cc|cc} 0.20 & 0.10 & 1 & 0 \\ 0.02 & 0.06 & 0 & 1 \end{array}\right] \begin{array}{l} 5R_1 \to R_1 \\ 50R_2 \to R_2 \end{array} \sim \left[\begin{array}{cc|cc} 1 & 0.5 & 5 & 0 \\ 1 & 3 & 0 & 50 \end{array}\right] (-1)R_1 + R_2 \to R_2 \sim \left[\begin{array}{cc|cc} 1 & 0.5 & 5 & 0 \\ 0 & 2.5 & -5 & 50 \end{array}\right] 0.4R_2 \to R_2$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0.5 & 5 & 0 \\ 0 & 1 & -2 & 20 \end{array}\right] (-0.5)R_2 + R_1 \to R_1 \sim \left[\begin{array}{cc|cc} 1 & 0 & 6 & -10 \\ 0 & 1 & -2 & 20 \end{array}\right]$$

Hence $A^{-1} = \begin{bmatrix} 6 & -10 \\ -2 & 20 \end{bmatrix}$ Check: $A^{-1}A = \begin{bmatrix} 6 & -10 \\ -2 & 20 \end{bmatrix} \begin{bmatrix} 0.20 & 0.10 \\ 0.02 & 0.06 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

We can now solve the system as

$$\underset{X}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{A^{-1}}{\begin{bmatrix} 6 & -10 \\ -2 & 20 \end{bmatrix}} \underset{B}{\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}}$$

For Diet 1, $k_1 = 20$ and $k_2 = 6$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 & -10 \\ -2 & 20 \end{bmatrix} \begin{bmatrix} 20 \\ 6 \end{bmatrix} = \begin{bmatrix} 60 \\ 80 \end{bmatrix}$$

Diet 1: 60 ounces Mix A and 80 ounces Mix B

For Diet 2, $k_1 = 10$ and $k_2 = 4$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 & -10 \\ -2 & 20 \end{bmatrix} \begin{bmatrix} 10 \\ 4 \end{bmatrix} = \begin{bmatrix} 20 \\ 60 \end{bmatrix}$$

Diet 2: 20 ounces Mix A and 60 ounces Mix B

For Diet 3, $k_1 = 10$ and $k_2 = 6$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 & -10 \\ -2 & 20 \end{bmatrix} \begin{bmatrix} 10 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \end{bmatrix}$$

Diet 3: 0 ounces Mix A and 100 ounces Mix B

Section 10–5

1. One is a matrix, the other is a determinant.

3. A minor is a determinant obtained by crossing out row i and column j of a larger determinant. A cofactor is $(-1)^{i+j}$ times the minor of the element in row i, column j.

5. The system does not have a unique solution; it is either inconsistent or dependent.

7. Yes, Cramer's method can be used, but doing it by hand is tedious.

9. $\begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix} = 5 \cdot 3 - 2 \cdot 4 = 7$

11. $\begin{vmatrix} 3 & -7 \\ -5 & 6 \end{vmatrix} = 3 \cdot 6 - (-5)(-7) = -17$

13. $\begin{vmatrix} 4.3 & -1.2 \\ -5.1 & 3.7 \end{vmatrix} = (4.3)(3.7) - (-5.1)(-1.2) = 9.79$

15. $D = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1;\ x = \dfrac{\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}}{D} = \dfrac{5}{1} = 5;\ y = \dfrac{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}}{D} = \dfrac{-2}{1} = -2$

17. $D = \begin{vmatrix} 2 & 1 \\ 5 & 3 \end{vmatrix} = 1;\ x = \dfrac{\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}}{D} = \dfrac{1}{1} = 1;\ y = \dfrac{\begin{vmatrix} 2 & 1 \\ 5 & 2 \end{vmatrix}}{D} = \dfrac{-1}{1} = -1$

19. $D = \begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} = 5;\ x = \dfrac{\begin{vmatrix} -3 & -1 \\ 3 & 3 \end{vmatrix}}{D} = \dfrac{-6}{5} = -\dfrac{6}{5};\ y = \dfrac{\begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix}}{D} = \dfrac{3}{5}$

21. $D = \begin{vmatrix} 4 & -3 \\ 3 & 2 \end{vmatrix} = 17;\ x = \dfrac{\begin{vmatrix} 4 & -3 \\ -2 & 2 \end{vmatrix}}{D} = \dfrac{2}{17};\ y = \dfrac{\begin{vmatrix} 4 & 4 \\ 3 & -2 \end{vmatrix}}{D} = \dfrac{-20}{17} = -\dfrac{20}{17}$

23. $\begin{vmatrix} 5 & -1 & -3 \\ 3 & 4 & 6 \\ 0 & -2 & 8 \end{vmatrix} = \begin{vmatrix} 4 & 6 \\ -2 & 8 \end{vmatrix}$

25. $\begin{vmatrix} 5 & -1 & -3 \\ 3 & 4 & 6 \\ 0 & -2 & 8 \end{vmatrix} = \begin{vmatrix} 5 & -1 \\ 0 & -2 \end{vmatrix}$

27. $(-1)^{1+1}\begin{vmatrix} 4 & 6 \\ -2 & 8 \end{vmatrix} = (-1)^2[4 \cdot 8 - (-2)6] = 44$

29. $(-1)^{2+3}\begin{vmatrix} 5 & -1 \\ 0 & -2 \end{vmatrix} = (-1)^5[5(-2) - 0(-1)] = 10$

31. We expand by row 1

$$\begin{vmatrix} 1 & 0 & 0 \\ -2 & 4 & 3 \\ 5 & -2 & 1 \end{vmatrix} = a_{11}(\text{cofactor of } a_{11}) + a_{12}(\text{cofactor of } a_{12}) + a_{13}(\text{cofactor of } a_{13})$$

$$= 1(-1)^{1+1}\begin{vmatrix} 4 & 3 \\ -2 & 1 \end{vmatrix} + 0(\ /\ ) + 0(\ \backslash\ )$$

It is unnecessary to evaluate these since they are multiplied by 0.

$$= (-1)^2[4 \cdot 1 - (-2)3] = 10$$

33. We expand by column 1

$$\begin{vmatrix} 0 & 1 & 5 \\ 3 & -7 & 6 \\ 0 & -2 & -3 \end{vmatrix} = a_{11}(\text{cofactor of } a_{11}) + a_{21}(\text{cofactor of } a_{21}) + a_{31}(\text{cofactor of } a_{31})$$

$$= 0(\ /\ ) + 3(-1)^{2+1}\begin{vmatrix} 1 & 5 \\ -2 & -3 \end{vmatrix} + 0(\ \backslash\ )$$

It is unnecessary to evaluate these since they are multiplied by 0.

$$= 3(-1)^3[1(-3) - (-2)5] = -21$$

Common Error: Neglecting the sign of the cofactor. The cofactor is often called the "signed" minor.

35. We expand by column 2

$$\begin{vmatrix} -1 & 2 & -3 \\ -2 & 0 & -6 \\ 4 & -3 & 2 \end{vmatrix} = a_{12}(\text{cofactor of } a_{12}) + a_{22}(\text{cofactor of } a_{22}) + a_{32}(\text{cofactor of } a_{32})$$

$$= 2(-1)^{1+2}\begin{vmatrix} -2 & -6 \\ 4 & 2 \end{vmatrix} + 0(\ /\ ) + (-3)(-1)^{3+2}\begin{vmatrix} -1 & -3 \\ -2 & -6 \end{vmatrix}$$

It is unnecessary to evaluate this since it's multiplied by zero.

$$= 2(-1)^3[(-2)2 - 4(-6)] + (-3)(-1)^5[(-1)(-6) - (-2)(-3)] = (-2)(20) + 3(0) = -40$$

37. We expand by the first row

$$\begin{vmatrix} 1 & 4 & 1 \\ 1 & 1 & -2 \\ 2 & 1 & -1 \end{vmatrix} = a_{11}(\text{cofactor of } a_{11}) + a_{12}(\text{cofactor of } a_{12}) + a_{13}(\text{cofactor of } a_{13})$$

$$= 1(-1)^{1+1}\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} + 4(-1)^{1+2}\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} + 1(-1)^{1+3}\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}$$

$$= (-1)^2[1(-1) - 1(-2)] + 4(-1)^3[1(-1) - 2(-2)] + (-1)^4[1\cdot 1 - 2\cdot 1] = 1 + (-12) + (-1) = -12$$

39. We expand by the first row

$$\begin{vmatrix} 1 & 4 & 3 \\ 2 & 1 & 6 \\ 3 & -2 & 9 \end{vmatrix} = a_{11}(\text{cofactor of } a_{11}) + a_{12}(\text{cofactor of } a_{12}) + a_{13}(\text{cofactor of } a_{13})$$

$$= 1(-1)^{1+1}\begin{vmatrix} 1 & 6 \\ -2 & 9 \end{vmatrix} + 4(-1)^{1+2}\begin{vmatrix} 2 & 6 \\ 3 & 9 \end{vmatrix} + 3(-1)^{1+3}\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix}$$

$$= (-1)^2[1\cdot 9 - (-2)6] + 4(-1)^3[2\cdot 9 - 3\cdot 6] + 3(-1)^4[2(-2) - 1\cdot 3] = 21 + 0 - 21 = 0$$

41. $D = \begin{vmatrix} 0.9925 & -0.9659 \\ 0.1219 & 0.2588 \end{vmatrix} = 0.37460$

$$x = \frac{\begin{vmatrix} 0 & -0.9659 \\ 2{,}500 & 0.2588 \end{vmatrix}}{D} = \frac{2{,}414.75}{0.37460} = 6{,}400 \text{ to two significant digits}$$

$$y = \frac{\begin{vmatrix} 0.9925 & 0 \\ 0.1219 & 2{,}500 \end{vmatrix}}{D} = \frac{2{,}481.25}{0.37460} = 6{,}600 \text{ to two significant digits}$$

43. $D = \begin{vmatrix} 0.9954 & -0.9942 \\ 0.0958 & 0.1080 \end{vmatrix} = 0.20275$

$$x = \frac{\begin{vmatrix} 0 & -0.9942 \\ 155 & 0.1080 \end{vmatrix}}{D} = \frac{154.10}{0.20275} = 760 \text{ to two significant digits}$$

$$y = \frac{\begin{vmatrix} 0.9954 & 0 \\ 0.0958 & 155 \end{vmatrix}}{D} = \frac{154.29}{0.20275} = 760 \text{ to two significant digits}$$

45. $D = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 1$; $x = \dfrac{\begin{vmatrix} 0 & 1 & 0 \\ -5 & 2 & 1 \\ -3 & 0 & 1 \end{vmatrix}}{D} = \dfrac{2}{1} = 2$; $y = \dfrac{\begin{vmatrix} 1 & 0 & 0 \\ 0 & -5 & 1 \\ -1 & -3 & 1 \end{vmatrix}}{D} = \dfrac{-2}{1} = -2$; $z = \dfrac{\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & -5 \\ -1 & 0 & -3 \end{vmatrix}}{D} = \dfrac{-1}{1} = -1$

47. $D = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 1 \end{vmatrix} = 3$; $x = \dfrac{\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & -1 & 1 \end{vmatrix}}{D} = \dfrac{4}{3}$; $y = \dfrac{\begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}}{D} = \dfrac{-1}{3} = -\dfrac{1}{3}$; $z = \dfrac{\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{vmatrix}}{D} = \dfrac{2}{3}$

49. $D = \begin{vmatrix} 0 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & -3 & 0 \end{vmatrix} = 3$; $x = \dfrac{\begin{vmatrix} -1 & 3 & 1 \\ 3 & 0 & 2 \\ -2 & -3 & 0 \end{vmatrix}}{D} = \dfrac{-27}{3} = -9$; $y = \dfrac{\begin{vmatrix} 0 & -1 & 1 \\ 1 & 3 & 2 \\ 1 & -2 & 0 \end{vmatrix}}{D} = \dfrac{-7}{3} = -\dfrac{7}{3}$;

$$z = \frac{\begin{vmatrix} 0 & 3 & -1 \\ 1 & 0 & 3 \\ 1 & -3 & -2 \end{vmatrix}}{D} = \frac{18}{3} = 6$$

51. $D = \begin{vmatrix} 0 & 2 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 2 \end{vmatrix} = -6$; $x = \dfrac{\begin{vmatrix} -3 & 2 & -1 \\ 2 & -1 & -1 \\ 4 & -1 & 2 \end{vmatrix}}{D} = \dfrac{-9}{-6} = \dfrac{3}{2}$; $y = \dfrac{\begin{vmatrix} 0 & -3 & -1 \\ 1 & 2 & -1 \\ 1 & 4 & 2 \end{vmatrix}}{D} = \dfrac{7}{-6} = -\dfrac{7}{6}$;

$$z = \frac{\begin{vmatrix} 0 & 2 & -3 \\ 1 & -1 & 2 \\ 1 & -1 & 4 \end{vmatrix}}{D} = \frac{-4}{-6} = \frac{2}{3}$$

53. Compute the coefficient determinant:

$D = \begin{vmatrix} a & 3 \\ 2 & 4 \end{vmatrix} = 4a - 3(2) = 4a - 6$

If $D \neq 0$, there is a unique solution: $4a - 6 = 0$

$$4a = 6$$
$$a = \frac{6}{4} = \frac{3}{2}$$

So if $a \neq \frac{3}{2}$ there is one solution. If $a = \frac{3}{2}$ we need to use Gauss–Jordan elimination to determine the nature of the solutions after plugging in $\frac{3}{2}$ for a.

$$\left[\begin{array}{cc|c} \frac{3}{2} & 3 & b \\ 2 & 4 & 5 \end{array}\right] \tfrac{2}{3}R_1 \to R_1 \sim \left[\begin{array}{cc|c} 1 & 2 & \frac{2b}{3} \\ 2 & 4 & 5 \end{array}\right] -2R_1 + R_2 \to R_2 \sim \left[\begin{array}{cc|c} 1 & 2 & \frac{2b}{3} \\ 0 & 0 & -\frac{4b}{3}+5 \end{array}\right]$$

If the bottom row is all zeros, there are infinitely many solutions. If the bottom row has zero in the third position, there is no solution. We need to know when $\frac{-4b}{3} + 5 = 0$.

$$\frac{-4b}{3} + 5 = 0$$
$$-4b + 15 = 0$$
$$-4b = -15$$
$$b = \frac{15}{4}$$

So there are infinitely many solutions if $a = \frac{3}{2}$ and $b = \frac{15}{4}$, and no solutions if $a = \frac{3}{2}$ and $b \neq \frac{15}{4}$.

55. $x = \dfrac{\begin{vmatrix} -3 & -3 & 1 \\ -11 & 3 & 2 \\ 3 & -1 & -1 \end{vmatrix}}{\begin{vmatrix} 2 & -3 & 1 \\ -4 & 3 & 2 \\ 1 & -1 & -1 \end{vmatrix}} = \dfrac{20}{5} = 4$

57. $y = \dfrac{\begin{vmatrix} 12 & 5 & 11 \\ 15 & -13 & -9 \\ 5 & 0 & 2 \end{vmatrix}}{\begin{vmatrix} 12 & -14 & 11 \\ 15 & 7 & -9 \\ 5 & -3 & 2 \end{vmatrix}} = \dfrac{28}{14} = 2$

59. $z = \dfrac{\begin{vmatrix} 3 & -4 & 18 \\ -9 & 8 & -13 \\ 5 & -7 & 33 \end{vmatrix}}{\begin{vmatrix} 3 & -4 & 5 \\ -9 & 8 & 7 \\ 5 & -7 & 10 \end{vmatrix}} = \dfrac{5}{2}$

61. $\begin{vmatrix} 2 & 6 & -1 & 2 & 6 \\ 5 & 3 & -7 & 5 & 3 \\ -4 & -2 & 1 & -4 & -2 \end{vmatrix}$

$2\cdot3\cdot1 + 6(-7)(-4) + (-1)(5)(-2) - (-4)(3)(-1) - (-2)(-7)2 - 1\cdot5\cdot6 = 6 + 168 + 10 - 12 - 28 - 30 = 114$

63. False. $\begin{vmatrix} 10 & 10 \\ 0 & 0 \end{vmatrix}$ is a counterexample.

65. True. Expanding $\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{vmatrix}$ by the first column, we obtain successively

$$a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ 0 & a_{33} & a_{34} \\ 0 & 0 & a_{44} \end{vmatrix} = a_{11}a_{22}\begin{vmatrix} a_{33} & a_{34} \\ 0 & a_{44} \end{vmatrix} = a_{11}a_{22}a_{33}a_{44}.$$

Similarly for the determinant of an $n \times n$ upper triangular matrix, we would obtain $a_{11}a_{22}a_{33} \cdot \ldots \cdot a_{nn}$ as proposed.

67. Expanding by the first column

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}(-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}(-1)^{3+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}(a_{12}a_{33} - a_{32}a_{13}) + a_{31}(a_{12}a_{23} - a_{22}a_{13})$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13}$$

(1) (2) (3) (4) (5) (6)

Expanding by the third row

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{31}(-1)^{3+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + a_{32}(-1)^{3+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33}(-1)^{3+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{31}(a_{12}a_{23} - a_{13}a_{22}) - a_{32}(a_{11}a_{23} - a_{13}a_{21}) + a_{33}(a_{11}a_{22} - a_{12}a_{21})$$

$$= a_{31}a_{12}a_{23} - a_{31}a_{13}a_{22} - a_{32}a_{11}a_{23} + a_{32}a_{13}a_{21} + a_{33}a_{11}a_{22} - a_{33}a_{12}a_{21}$$

(5) (6) (2) (4) (1) (3)

Comparing the two expressions, with the aid of the numbers under the terms, shows that the expressions are the same.

69. $A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}$

We calculate $AB = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2(-1)+3\cdot 2 & 2\cdot 3+3\cdot 1 \\ 1(-1)+(-2)2 & 1\cdot 3+(-2)\cdot 1 \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ -5 & 1 \end{bmatrix}$

$$\det(AB) = \begin{vmatrix} 4 & 9 \\ -5 & 1 \end{vmatrix} = 4 \cdot 1 - (-5)9 = 49$$

$$\det A = \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = 2(-2) - 1 \cdot 3 = -7$$

$$\det B = \begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} = (-1)1 - 2 \cdot 3 = -7$$

Thus, $\det A \det B = (-7)(-7) = 49 = \det(AB)$

71. (A) $D = \begin{vmatrix} 1 & -4 & 9 \\ 4 & -1 & 6 \\ 1 & -1 & 3 \end{vmatrix} = 1(-1)^{1+1}\begin{vmatrix} -1 & 6 \\ -1 & 3 \end{vmatrix} + 4(-1)^{2+1}\begin{vmatrix} -4 & 9 \\ -1 & 3 \end{vmatrix} + 1(-1)^{3+1}\begin{vmatrix} -4 & 9 \\ -1 & 6 \end{vmatrix}$

$$= 1(-1)^2[(-1)3-(-1)6]+4(-1)^3[(-4)3-(-1)9]+1(-1)^4[(-4)6-(-1)9] = 3 + 12 - 15 = 0$$

Since $D = 0$, the system either has no solution or infinitely many. Since $x = 0$, $y = 0$, $z = 0$ is a solution, the second case must hold.

(B) $D = \begin{vmatrix} 3 & -1 & 3 \\ 5 & 5 & -9 \\ -2 & 1 & -3 \end{vmatrix} = -6 \neq 0$; $x = 0$, $y = 0$, $z = 0$ is the only solution.

73. (A) $R = xp + yq = (200 - 6p + 4q)p + (300 + 2p - 3q)q = 200p - 6p^2 + 4pq + 300q + 2pq - 3q^2$
$= 200p + 300q - 6p^2 + 6pq - 3q^2$

(B) Rewrite the demand equations as
$6p - 4q = 200 - x$
$-2p + 3q = 300 - y$

Apply Cramer's rule: $D = \begin{vmatrix} 6 & -4 \\ -2 & 3 \end{vmatrix} = 10$

$$p = \frac{\begin{vmatrix} 200-x & -4 \\ 300-y & 3 \end{vmatrix}}{D} = \frac{1800-3x-4y}{10} = -0.3x - 0.4y + 180$$

$$q = \frac{\begin{vmatrix} 6 & 200-x \\ -2 & 300-y \end{vmatrix}}{D} = \frac{2200-2x-6y}{10} = -0.2x - 0.6y + 220$$

Then

$R = xp + yq = x(-0.3x - 0.4y + 180) + y(-0.2x - 0.6y + 220)$
$= -0.3x^2 - 0.4xy + 180x - 0.2xy - 0.6y^2 + 220y$
$= 180x + 220y - 0.3x^2 - 0.6xy - 0.6y^2$

Note: Sections 6, 7, 8 of this chapter are not printed in the text. They are available online. Solutions follow.

Section 10–6

1. Unless each equation is linear, that is, has the form $a_1x_1 + a_2x_2 + \ldots a_nx_n = b$, the system is non linear.

3. Substitution would be preferable. Solve the first equation for x in terms of y (or y in terms of x), substitute into the second equation, solve the single–variable equation that results, and plug any solutions into the first equation, to find the other variable.

5.
$$x^2 + y^2 = 169$$
$$x = -12$$
$$(-12)^2 + y^2 = 169$$
$$y^2 = 25$$
$$y = \pm 5$$

Solution: $(-12, 5)$, $(-12, -5)$

Check: $-12 \stackrel{\checkmark}{=} -12$

$(-12)^2 + (\pm 5)^2 \stackrel{\checkmark}{=} 169$

7.
$$8x^2 - y^2 = 16$$
$$y = 2x$$

Substitute y from the second equation into the first equation.
$$8x^2 - (2x)^2 = 16$$
$$8x^2 - 4x^2 = 16$$
$$4x^2 = 16$$
$$x^2 = 4$$
$$x = \pm 2$$

For $x = 2$ For $x = -2$
$y = 2(2)$ $y = 2(-2)$
$y = 4$ $y = -4$

Solutions: $(2, 4)$, $(-2, -4)$

Check:

For $(2, 4)$: $4 \stackrel{\checkmark}{=} 2\cdot 2$, $8(2)^2 - 4^2 \stackrel{\checkmark}{=} 16$

For $(-2, -4)$: $-4 \stackrel{\checkmark}{=} 2(-2)$, $8(-2)^2 - (-4)^2 \stackrel{\checkmark}{=} 16$

9.
$$3x^2 - 2y^2 = 25$$
$$x + y = 0$$

Solve for y in the first degree equation: $y = -x$
Substitute into the second degree equation.
$$3x^2 - 2(-x)^2 = 25$$
$$x^2 = 25$$
$$x = \pm 5$$

For $x = 5$ For $x = -5$
$y = -5$ $y = 5$

Solutions: $(5, -5)$, $(-5, 5)$

Check:

For $(5, -5)$: $5 + (-5) \stackrel{\checkmark}{=} 0$, $3(5)^2 - 2(-5)^2 \stackrel{\checkmark}{=} 25$

For $(-5, 5)$: $(-5) + 5 \stackrel{\checkmark}{=} 0$, $3(-5)^2 - 2(5)^2 \stackrel{\checkmark}{=} 25$

From this point on we will not show the checking steps for lack of space. The student should perform these checking steps, however.

11.
$$y^2 = x$$
$$x - 2y = 2$$

Solve for x in the first degree equation.
$x = 2y + 2$
Substitute into the second degree equation.
$$y^2 = 2y + 2$$
$$y^2 - 2y - 2 = 0$$
$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad a = 1, b = -2, c = -2$$
$$y = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)}$$
$$y = \frac{2 \pm \sqrt{12}}{2}$$
$$y = 1 \pm \sqrt{3}$$

For $y = 1 + \sqrt{3}$ For $y = 1 - \sqrt{3}$
$x = 2(1 + \sqrt{3}) + 2$ $x = 2(1 - \sqrt{3}) + 2$
$x = 4 + 2\sqrt{3}$ $x = 4 - 2\sqrt{3}$

Solutions: $(4 + 2\sqrt{3}, 1 + \sqrt{3})$, $(4 - 2\sqrt{3}, 1 - \sqrt{3})$

13.
$$2x^2 + y^2 = 24$$
$$x^2 - y^2 = -12$$

Solve using elimination by addition. Adding, we obtain:
$$3x^2 = 12$$
$$x^2 = 4$$
$$x = \pm 2$$

For $x = 2$ For $x = -2$
$4 - y^2 = -12$ $4 - y^2 = -12$
$-y^2 = -16$ Similarly
$y^2 = 16$ $y = \pm 4$
$y = \pm 4$

Solutions: $(2, 4)$, $(2, -4)$, $(-2, 4)$, $(-2, -4)$

15. $x^2 + y^2 = 10$
$16x^2 + y^2 = 25$

Solve using elimination by addition.
Multiply the top equation by –1 and add.

$$-x^2 - y^2 = -10$$
$$16x^2 + y^2 = 25$$
$$15x^2 = 15$$
$$x^2 = 1$$
$$x = \pm 1$$

For $x = 1$: $1 + y^2 = 10$, $y^2 = 9$, $y = \pm 3$

For $x = -1$: $1 + y^2 = 10$, $y = \pm 3$

Solutions: (1, 3), (1, –3), (–1, 3), (–1, –3)

17. $xy - 4 = 0$
$x - y = 2$

Solve for x in the first degree equation.
$x = y + 2$
Substitute into the second degree equation

$$(y + 2)y - 4 = 0$$
$$y^2 + 2y - 4 = 0$$
$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad a = 1, b = 2, c = -4$$
$$y = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-4)}}{2(1)}$$
$$y = \frac{-2 \pm \sqrt{20}}{2}$$
$$y = -1 \pm \sqrt{5}$$

For $y = -1 + \sqrt{5}$: $x = -1 + \sqrt{5} + 2$, $x = 1 + \sqrt{5}$

For $y = -1 - \sqrt{5}$: $x = -1 - \sqrt{5} + 2$, $x = 1 - \sqrt{5}$

Solutions: $(1 + \sqrt{5}, -1 + \sqrt{5})$, $(1 - \sqrt{5}, -1 - \sqrt{5})$

19. $x^2 + 2y^2 = 6$
$xy = 2$

Solve for y in the second equation
$y = \frac{2}{x}$
Substitute into the first equation

$$x^2 + 2\left(\frac{2}{x}\right)^2 = 6$$
$$x^2 + \frac{8}{x^2} = 6 \quad x \neq 0$$
$$x^2 \cdot x^2 + x^2 \cdot \frac{8}{x^2} = 6x^2$$
$$x^4 + 8 = 6x^2$$
$$x^4 - 6x^2 + 8 = 0$$
$$(x^2 - 2)(x^2 - 4) = 0$$
$$(x - \sqrt{2})(x + \sqrt{2})(x - 2)(x + 2) = 0$$
$$x = \sqrt{2}, -\sqrt{2}, 2, -2$$

For $x = \sqrt{2}$: $y = \frac{2}{\sqrt{2}}$, $y = \sqrt{2}$

For $x = -\sqrt{2}$: $y = -\frac{2}{\sqrt{2}}$, $y = -\sqrt{2}$

For $x = 2$: $y = \frac{2}{2}$, $y = 1$

For $x = -2$: $y = \frac{2}{-2}$, $y = -1$

Solutions: $(\sqrt{2}, \sqrt{2})$, $(-\sqrt{2}, -\sqrt{2})$, (2, 1), (–2, –1)

21. $2x^2 + 3y^2 = -4$
$4x^2 + 2y^2 = 8$

Solve using elimination by addition.
Multiply the second equation by $-\frac{1}{2}$ and add.

$$2x^2 + 3y^2 = -4$$
$$-2x^2 - y^2 = -4$$
$$2y^2 = -8$$
$$y^2 = -4$$
$$y = \pm 2i$$

For $y = 2i$: $2x^2 + 3(2i)^2 = -4$, $2x^2 - 12 = -4$, $2x^2 = 8$, $x^2 = 4$, $x = \pm 2$

For $y = -2i$: $2x^2 + 3(-2i)^2 = -4$, $2x^2 - 12 = -4$, Similarly $x = \pm 2$

Solutions: $(2, 2i)$, $(-2, 2i)$, $(2, -2i)$, $(-2, -2i)$

23. $x^2 - y^2 = 2$
$y^2 = x$

Substitute y^2 from the second equation into the first equation.

$$x^2 - x = 2$$
$$x^2 - x - 2 = 0$$
$$(x - 2)(x + 1) = 0$$
$$x = 2, -1$$

For $x = 2$: $y^2 = 2$, $y = \pm\sqrt{2}$

For $x = -1$: $y^2 = -1$, $y = \pm i$

Solutions: $(2, \sqrt{2})$, $(2, -\sqrt{2})$, $(-1, i)$, $(-1, -i)$

25. $x^2 + y^2 = 9$

27. $x^2 - y^2 = 3$

$x^2 = 9 - 2y$

Substitute x^2 from the second equation into the first equation.

$$9 - 2y + y^2 = 9$$
$$y^2 - 2y = 0$$
$$y(y - 2) = 0$$
$$y = 0, 2$$

For $y = 0$: $x^2 = 9 - 2(0)$, $x^2 = 9$, $x = \pm 3$

For $y = 2$: $x^2 = 9 - 2(2)$, $x^2 = 5$, $x = \pm\sqrt{5}$

Solutions: $(3, 0), (-3, 0), (\sqrt{5}, 2), (-\sqrt{5}, 2)$

$xy = 2$

Solve for y in the second equation.

$$y = \frac{2}{x}$$

Substitute into the first equation:

$$x^2 - \left(\frac{2}{x}\right)^2 = 3$$
$$x^2 - \frac{4}{x^2} = 3 \quad x \neq 0$$
$$x^4 - 4 = 3x^2$$
$$x^4 - 3x^2 - 4 = 0$$
$$(x^2 - 4)(x^2 + 1) = 0$$

$x^2 - 4 = 0$, $x^2 = 4$, $x = \pm 2$

$x^2 + 1 = 0$, $x^2 = -1$, $x = \pm i$

For $x = 2$: $y = \frac{2}{2}$, $y = 1$

For $x = -2$: $y = \frac{2}{-2}$, $y = -1$

For $x = i$: $y = \frac{2}{i}$, $y = -2i$

For $x = -i$: $y = \frac{2}{-i}$, $y = 2i$

Solutions: $(2, 1), (-2, -1), (i, -2i), (-i, 2i)$

29. $y = 5 - x^2$

$y = 2 - 2x$

Substitute y from the first equation into the second equation.

$$5 - x^2 = 2 - 2x$$
$$0 = x^2 - 2x - 3$$
$$0 = (x - 3)(x + 1)$$
$$x = 3, -1$$

For $x = 3$: $y = 2 - 2(3)$, $y = -4$

For $x = -1$: $y = 2 - 2(-1)$, $y = 4$

Solutions: $(3, -4), (-1, 4)$

31. $y = x^2 - x$

$y = 2x$

Substitute y from the first equation into the second equation.

$$x^2 - x = 2x$$
$$x^2 - 3x = 0$$
$$x(x - 3) = 0$$
$$x = 0, 3$$

For $x = 0$: $y = 2(0)$, $y = 0$

For $x = 3$: $y = 2(3)$, $y = 6$

Solutions: $(0, 0), (3, 6)$

33. $y = x^2 - 6x + 9$

$y = 5 - x$

Substitute y from the first equation into the second equation.

$$x^2 - 6x + 9 = 5 - x$$
$$x^2 - 5x + 4 = 0$$
$$(x - 1)(x - 4) = 0$$
$$x = 1, 4$$

For $x = 1$: $y = 5 - 1$, $y = 4$

For $x = 4$: $y = 5 - 4$, $y = 1$

Solutions: $(1, 4), (4, 1)$

35. $y = 8 + 4x - x^2$

$y = x^2 - 2x$

Substitute y from the first equation into the second equation.

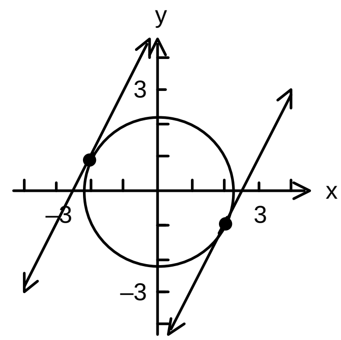
$$8 + 4x - x^2 = x^2 - 2x$$
$$0 = 2x^2 - 6x - 8$$
$$0 = x^2 - 3x - 4$$
$$0 = (x - 4)(x + 1)$$
$$x = 4, -1$$

For $x = 4$: $y = 4^2 - 2(4)$, $y = 8$

For $x = -1$: $y = (-1)^2 - 2(-1)$, $y = 3$

Solutions: $(4, 8), (-1, 3)$

37. (A) The lines are tangent to the circle.



(B) To find values of b such that

$$x^2 + y^2 = 5$$
$$2x - y = b$$

has exactly one solution, we solve the system for arbitrary b.
Solve for y in the second equation.
$y = 2x - b$
Substitute into the first equation:

$$x^2 + (2x - b)^2 = 5$$
$$x^2 + 4x^2 - 4bx + b^2 = 5$$
$$5x^2 - 4bx + b^2 - 5 = 0$$

This quadratic equation will have one solution if the discriminant
$B^2 - 4AC = (-4b)^2 - 4(5)(b^2 - 5)$ is equal to 0.
This will occur when

$$16b^2 - 20b^2 + 100 = 0$$
$$-4b^2 + 100 = 0$$
$$b^2 = 25$$
$$b = \pm 5$$

Consider $b = 5$
Then the solution of the system

$$x^2 + y^2 = 5$$
$$2x - y = 5$$

will be given by solving
$5x^2 - 4bx + b^2 - 5 = 0$
for $b = 5$.

$$5x^2 - 4 \cdot 5x + 5^2 - 5 = 0$$
$$5x^2 - 20x + 20 = 0$$
$$5(x - 2)^2 = 0$$
$$x - 2 = 0$$
$$x = 2$$

Since $2x - y = 5$
$2 \cdot 2 - y = 5$
$y = -1$
The intersection point is $(2, -1)$ for $b = 5$.

Consider $b = -5$
Then the solution of the system

$$x^2 + y^2 = 5$$
$$2x - y = -5$$

will be given by solving
$5x^2 - 4bx + b^2 - 5 = 0$
for $b = -5$.

$$5x^2 - 4(-5)x + (-5)^2 - 5 = 0$$
$$5x^2 + 20x + 20 = 0$$
$$5(x + 2)^2 = 0$$
$$x + 2 = 0$$
$$x = -2$$

Since $2x - y = -5$
$2(-2) - y = -5$
$y = 1$
The intersection point is $(-2, 1)$ for $b = -5$.

(C) The line $x + 2y = 0$ is perpendicular to all the lines in the family and intersects the circle at the intersection points found in part B, since this line passes through the center of the circle and thus includes a diameter of the circle, which is perpendicular to the tangent line at their mutual point of intersection with the circle. Solving the system $x^2 + y^2 = 5$, $x + 2y = 0$ would determine the intersection points.

39. $2x + 5y + 7xy = 8$

41. $x^2 - 2xy + y^2 = 1$

$xy - 3 = 0$

Solve for y in the second equation.

$xy = 3$

$y = \frac{3}{x}$

Substitute into the first equation.

$$2x + 5\left(\frac{3}{x}\right) + 7x\left(\frac{3}{x}\right) = 8$$

$$2x + \frac{15}{x} + 21 = 8 \quad x \neq 0$$

$$2x^2 + 15 + 21x = 8x$$

$$2x^2 + 13x + 15 = 0$$

$$(2x + 3)(x + 5) = 0$$

$$x = -\frac{3}{2}, -5$$

For $x = -\frac{3}{2}$

$y = 3 \div \left(-\frac{3}{2}\right)$

$y = -2$

For $x = -5$

$y = \frac{3}{-5}$

$y = -\frac{3}{5}$

Solutions: $\left(-\frac{3}{2}, -2\right)$, $\left(-5, -\frac{3}{5}\right)$

$x - 2y = 2$

Solve for x in terms of y in the first–degree equation.

$x = 2y + 2$

Substitute into the second–degree equation.

$$(2y + 2)^2 - 2(2y + 2)y + y^2 = 1$$

$$4y^2 + 8y + 4 - 4y^2 - 4y + y^2 = 1$$

$$y^2 + 4y + 3 = 0$$

$$(y + 1)(y + 3) = 0$$

$$y = -1, -3$$

For $y = -1$

$x = 2(-1) + 2$

$= 0$

For $y = -3$

$x = 2(-3) + 2$

$= -4$

Solutions: $(0, -1)$, $(-4, -3)$

43. $2x^2 - xy + y^2 = 8$

$x^2 - y^2 = 0$

Factor the left side of the equation that has a zero constant term.

$(x - y)(x + y) = 0$

$x = y$ or $x = -y$

> **Common Error:**
> It is incorrect to replace $x^2 - y^2 = 0$ or $x^2 = y^2$ by $x = y$.
> This neglects the possibility $x = -y$.

Thus, the original system is equivalent to the two systems

$2x^2 - xy + y^2 = 8$ $2x^2 - xy + y^2 = 8$

$x = y$ $x = -y$

These systems are solved by substitution.

First system:

$$2x^2 - xy + y^2 = 8$$

$$x = y$$

$$2y^2 - yy + y^2 = 8$$

$$2y^2 = 8$$

$$y^2 = 4$$

$$y = \pm 2$$

Second system:

$$2x^2 - xy + y^2 = 8$$

$$x = -y$$

$$2(-y)^2 - (-y)y + y^2 = 8$$

$$2y^2 + y^2 + y^2 = 8$$

$$4y^2 = 8$$

$$y^2 = 2$$

$$y = \pm\sqrt{2}$$

For $y = 2$

$x = 2$

For $y = -2$

$x = -2$

For $y = \sqrt{2}$

$x = -\sqrt{2}$

For $y = -\sqrt{2}$

$x = \sqrt{2}$

Solutions: $(2, 2)$, $(-2, -2)$, $(-\sqrt{2}, \sqrt{2})$, $(\sqrt{2}, -\sqrt{2})$

45. $x^2 + xy - 3y^2 = 3$

$x^2 + 4xy + 3y^2 = 0$

Factor the left side of the equation that has a zero constant term.

$(x + y)(x + 3y) = 0$

$x = -y$ or $x = -3y$

Thus the original system is equivalent to the two systems

$$x^2 + xy - 3y^2 = 3 \qquad x^2 + xy - 3y^2 = 3$$
$$x = -y \qquad x = -3y$$

These systems are solved by substitution.

First system:

$$x^2 + xy - 3y^2 = 3$$
$$x = -y$$
$$(-y)^2 + (-y)y - 3y^2 = 3$$
$$y^2 - y^2 - 3y^2 = 3$$
$$-3y^2 = 3$$
$$y^2 = -1$$
$$y = \pm i$$

Second system:

$$x^2 + xy - 3y^2 = 3$$
$$x = -3y$$
$$(-3y)^2 + (-3y)y - 3y^2 = 3$$
$$9y^2 - 3y^2 - 3y^2 = 3$$
$$3y^2 = 3$$
$$y^2 = 1$$
$$y = \pm 1$$

For $y = i$, $x = -i$. For $y = -i$, $x = i$. For $y = 1$, $x = -3$. For $y = -1$, $x = 3$.

Solutions: $(-i, i)$, $(i, -i)$, $(-3, 1)$, $(3, -1)$

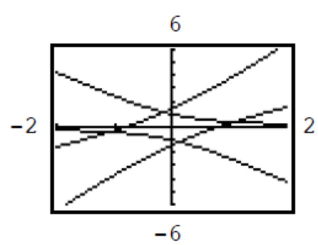
47. Before we can enter these equations in our graphing calculator, we must solve for y:

$$-x^2 + 2xy + y^2 = 1 \qquad 3x^2 - 4xy + y^2 = 2$$
$$y^2 + 2xy - 1 - x^2 = 0 \qquad y^2 - 4xy + 3x^2 - 2 = 0$$

Applying the quadratic formula to each equation, we have

$$y = \frac{-2x \pm \sqrt{4x^2 - 4(-1 - x^2)}}{2} \qquad y = \frac{4x \pm \sqrt{16x^2 - 4(3x^2 - 2)}}{2}$$
$$y = \frac{-2x \pm \sqrt{8x^2 + 4}}{2} \qquad y = \frac{4x \pm \sqrt{4x^2 + 8}}{2}$$
$$y = -x \pm \sqrt{2x^2 + 1} \qquad y = 2x \pm \sqrt{x^2 + 2}$$

Entering each of these four equations into a graphing calculator produces the graph shown at the right. Zooming in on the four intersection points, or using a built-in intersection routine (details omitted), yields $(-1.41, -0.82)$, $(-0.13, 1.15)$, $(0.13, -1.15)$, and $(1.41, 0.82)$ to two decimal places.



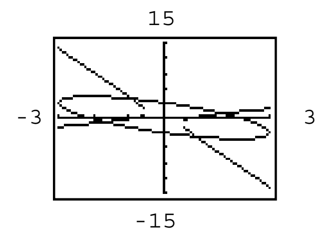
49. Before we can enter these equations in our graphing calculator, we must solve for y:

$$3x^2 - 4xy - y^2 = 2 \qquad 2x^2 + 2xy + y^2 = 9$$
$$y^2 + 4xy + 2 - 3x^2 = 0 \qquad y^2 + 2xy + 2x^2 - 9 = 0$$

Applying the quadratic formula to each equation, we have

$$y = \frac{-4x \pm \sqrt{16x^2 - 4(2 - 3x^2)}}{2} \qquad y = \frac{-2x \pm \sqrt{4x^2 - 4(2x^2 - 9)}}{2}$$
$$y = \frac{-4x \pm \sqrt{28x^2 - 8}}{2} \qquad y = \frac{-2x \pm \sqrt{36 - 4x^2}}{2}$$
$$y = -2x \pm \sqrt{7x^2 - 2} \qquad y = -x \pm \sqrt{9 - x^2}$$

Entering each of these four equations into a graphing calculator produces the graph shown at the right. Zooming in on the four intersection points, or using a built–in intersection routine (details omitted), yields (–1.66, –0.84), (–0.91, 3.77), (0.91, –3.77), and (1.66, 0.84) to two decimal places.



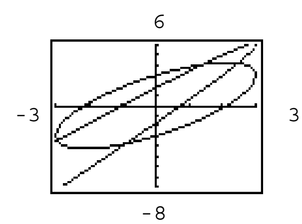
51. Before we can enter these equations in our graphing calculator, we must solve for y:

$$2x^2 - 2xy + y^2 = 9 \qquad\qquad 4x^2 - 4xy + y^2 + x = 3$$
$$y^2 - 2xy + 2x^2 - 9 = 0 \qquad\qquad y^2 - 4xy + 4x^2 + x - 3 = 0$$

Applying the quadratic formula to each equation, we have

$$y = \frac{2x \pm \sqrt{4x^2 - 4(2x^2 - 9)}}{2} \qquad\qquad y = \frac{4x \pm \sqrt{16x^2 - 4(4x^2 + x - 3)}}{2}$$
$$y = \frac{2x \pm \sqrt{36 - 4x^2}}{2} \qquad\qquad y = \frac{4x \pm \sqrt{12 - 4x}}{2}$$
$$y = x \pm \sqrt{9 - x^2} \qquad\qquad y = 2x \pm \sqrt{3 - x}$$

Entering each of these four equations into a graphing calculator produces the graph shown at the right. Zooming in on the four intersection points, or using a built–in intersection routine (details omitted), yields (–2.96, –3.47), (–0.89, –3.76), (1.39, 4.05), and (2.46, 4.18) to two decimal places.



53. Let x and y equal the two numbers. We have the system

$x + y = 3$

$xy = 1$

Solve the first equation for y in terms of x, then substitute into the second degree equation.

$$y = 3 - x$$
$$x(3 - x) = 1$$
$$3x - x^2 = 1$$
$$-x^2 + 3x - 1 = 0$$
$$x^2 - 3x + 1 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad a = 1, b = -3, c = 1$$
$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(1)}}{2(1)} = \frac{3 \pm \sqrt{5}}{2}$$

For $x = \frac{3+\sqrt{5}}{2}$

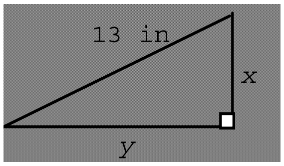
$$y = 3 - x = 3 - \frac{3+\sqrt{5}}{2} = \frac{6 - 3 - \sqrt{5}}{2} = \frac{3 - \sqrt{5}}{2}$$

For $x = \frac{3-\sqrt{5}}{2}$

$$y = 3 - x = 3 - \frac{3-\sqrt{5}}{2} = \frac{6 - 3 + \sqrt{5}}{2} = \frac{3 + \sqrt{5}}{2}$$

Thus the two numbers are $\frac{1}{2}(3 - \sqrt{5})$ and $\frac{1}{2}(3 + \sqrt{5})$.

55. Sketch a figure. Let x and y represent the lengths of the two legs.



From the Pythagorean Theorem we have $x^2 + y^2 = 13^2$

From the formula for the area of a triangle we have $\frac{1}{2}xy = 30$

Thus the system of equations is
$$x^2 + y^2 = 169$$
$$\frac{1}{2}xy = 30$$

Solve the second equation for y in terms of x, then substitute into the first equation.

$$xy = 60$$
$$y = \frac{60}{x}$$
$$x^2 + \left(\frac{60}{x}\right)^2 = 169$$
$$x^2 + \frac{3600}{x^2} = 169 \quad x \neq 0$$
$$x^4 + 3600 = 169x^2$$
$$x^4 - 169x^2 + 3600 = 0$$
$$(x^2 - 144)(x^2 - 25) = 0$$
$$(x - 12)(x + 12)(x - 5)(x + 5) = 0$$
$$x = \pm 12, \pm 5$$

Discarding the negative solutions, we have $x = 12$ or $x = 5$

For $x = 12$
$$y = \frac{60}{x}$$
$$y = 5$$

For $x = 5$
$$y = \frac{60}{x}$$
$$y = 12$$

The lengths of the legs are 5 inches and 12 inches.

57. Let x = width of screen.
y = height of screen.

From the Pythagorean Theorem, we have $x^2 + y^2 = (7.5)^2$

From the formula for the area of a rectangle we have $xy = 27$

Thus the system of equations is:
$$x^2 + y^2 = 56.25$$
$$xy = 27$$

Solve the second equation for y in terms of x, then substitute into the first equation.

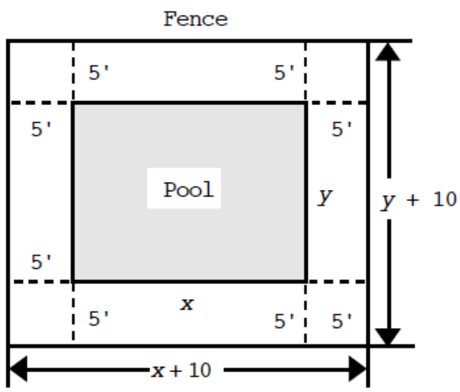
$$y = \frac{27}{x}$$
$$x^2 + \left(\frac{27}{x}\right)^2 = 56.25 \quad x \neq 0$$
$$x^2 + \frac{729}{x^2} = 56.25 \quad x \neq 0$$
$$x^4 + 729 = 56.25x^2$$
$$x^4 - 56.25x^2 + 729 = 0 \quad \text{quadratic in } x^2$$
$$x^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad a = 1,\ b = -56.25,\ c = 729$$
$$x^2 = \frac{-(-56.25) \pm \sqrt{(-56.25)^2 - 4(1)(729)}}{2(1)} = \frac{56.25 \pm 15.75}{2} = 36, 20.25$$
$$x = 6, 4.5 \text{ (discarding the negative solutions)}$$

For $x = 6$
$$y = \frac{27}{6} = 4.5$$

For $x = 4.5$
$$y = \frac{27}{4.5} = 6$$

The dimensions of the screen must be 6 inches by 4.5 inches.

59.



Redrawing and labeling the figure as shown, we have

Area of pool = 572

$xy = 572$

Area enclosed by fence = 1,152

$(x + 10)(y + 10) = 1{,}152$

We solve this system by solving for y in terms of x in the first equation, then substituting into the second equation.

$$y = \frac{572}{x}$$

$$(x + 10)\left(\frac{572}{x} + 10\right) = 1{,}152$$

$$572 + 10x + \frac{5{,}720}{x} + 100 = 1{,}152$$

$$10x + \frac{5{,}720}{x} - 480 = 0 \quad x \neq 0$$

$$10x^2 + 5{,}720 - 480x = 0$$

$$x^2 - 48x + 572 = 0$$

$$(x - 26)(x - 22) = 0$$

$$x = 26, 22$$

For $x = 26$

$$y = \frac{572}{26}$$

$$y = 22$$

For $x = 22$

$$y = \frac{572}{22}$$

$$y = 26$$

The dimensions of the pool are 22 feet by 26 feet.

61. Let x = average speed of Boat B

Then $x + 5$ = average speed of Boat A

Let y = time of Boat B, then $y - \frac{1}{2}$ = time of Boat A

Using Distance = rate × time, we have

$$75 = xy$$

$$75 = (x + 5)\left(y - \frac{1}{2}\right)$$

Note: The *faster* boat, A, has the *shorter* time. It is a common error to confuse the signs here. Another common error: if rates are expressed in miles per hour, then $y - 30$ is not the correct time for boat A. Times must be expressed in hours.

Solve the first equation for y in terms of x, then substitute into the second equation.

$$y = \frac{75}{x}$$

$$75 = (x + 5)\left(\frac{75}{x} - \frac{1}{2}\right)$$

$$75 = 75 - \frac{1}{2}x + \frac{375}{x} - \frac{5}{2}$$

$$0 = -\frac{1}{2}x + \frac{375}{x} - \frac{5}{2} \quad x \neq 0$$

$$2x(0) = 2x\left(-\frac{1}{2}x\right) + 2x\left(\frac{375}{x}\right) - 2x\left(\frac{5}{2}\right)$$

$$0 = -x^2 + 750 - 5x$$

$$x^2 + 5x - 750 = 0$$

$$(x - 25)(x + 30) = 0$$

$$x = 25, -30$$

Discarding the negative solution,

we have $x = 25$ mph = average speed of Boat B

$x + 5 = 30$ mph = average speed of Boat A

Section 10–7

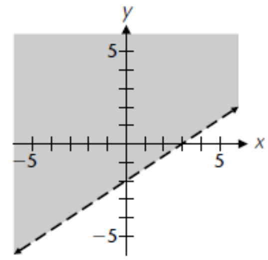
1. The graph of $y = mx + b$ is a straight line with slope m and y intercept b. The graph of $y < mx + b$ is a half-plane consisting of all points in the plane below the line. The graph of $y > mx + b$ is a half-plane consisting of all points in the plane above the line.

3. Variables representing numbers of real quantities (like tables or surfboards) cannot take on negative values. The graph of a linear inequality system with nonnegativity restrictions is in the first quadrant only.

5. Graph $2x - 3y = 6$ as a dashed line, since equality is not included in the original statement. The origin is a suitable test point.

$$2x - 3y < 6$$
$$2(0) - 3(0) = 0 < 6$$

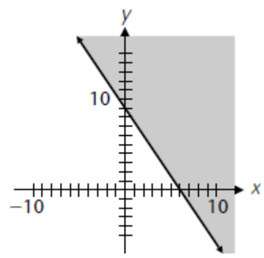
Hence (0, 0) is in the solution set.
The graph is the half–plane containing (0, 0).



7. Graph $3x + 2y = 18$ as a solid line, since equality is included in the original statement. The origin is a suitable test point.

$$3(0) + 2(0) = 0 \not\geq 18$$

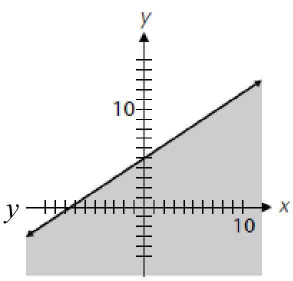
Hence (0, 0) is not in the solution set.
The graph is the line $3x + 2y = 18$ and the half–plane not containing the origin.



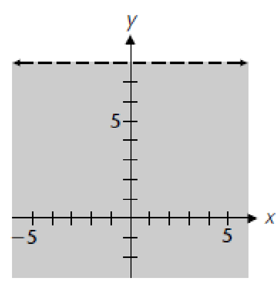
9. Graph $y = \frac{2}{3}x + 5$ as a solid line, since equality is included in the original statement. The origin is a suitable test point.

$$0 \overset{?}{\le} (0) + 5$$
$$0 \le 5$$

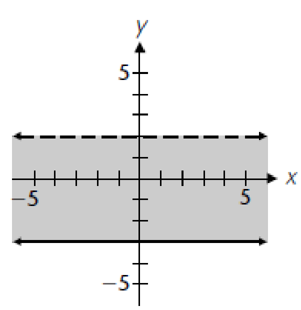
Hence (0, 0) is in the solution set. The graph is the line $y = \frac{2}{3}x + 5$ and the half–plane containing the origin.



11. Graph $y = 8$ as a dashed line, since equality is not included in the original statement. Clearly the graph consists of all points whose y–coordinates are less than 8, that is, the lower half-plane.



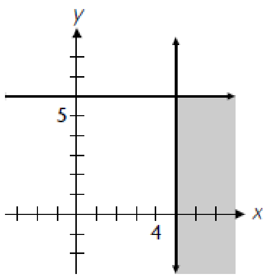
13. This system is equivalent to the system
$y \ge -3$
$y < 2$
and its graph is the intersection of the graphs of these inequalities.



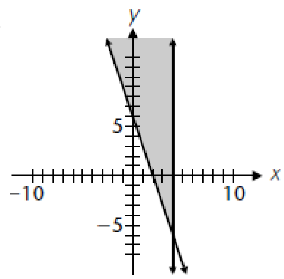
15. $x + 2y \leq 8$
$3x - 2y \geq 0$
Choose a suitable test point that lies on neither line, for example, (2, 0).
$2 + 2(0) = 2 \leq 8$ Hence, the solution region is *below* the graph of $x + 2y = 8$.
$3(2) - 2(0) = 6 \geq 0$ Hence, the solution region is *below* the graph of $3x - 2y = 0$.
Thus the solution region is region IV in the diagram.

17. $x + 2y \geq 8$
$3x - 2y \geq 0$
Choose a suitable test point that lies on neither line, for example, (2, 0).
$2 + 2(0) = 2 \not\geq 8$ Hence, the solution region is *above* the graph of $x + 2y = 8$.
$3(2) - 2(0) = 6 \geq 0$ Hence, the solution region is *below* the graph of $3x - 2y = 0$.
Thus the solution region is region I in the diagram.

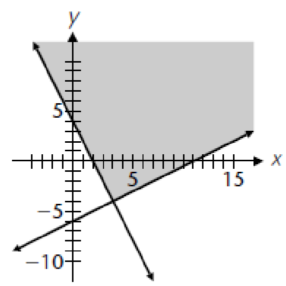
19.



21.



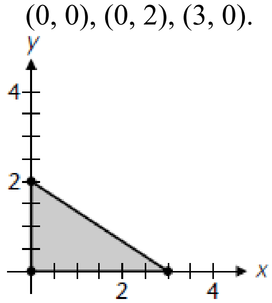
23.



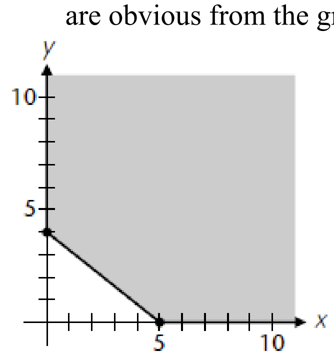
25. Choose a suitable test point that lies on none of the lines, say (5, 1).
$5 + 3(1) = 8 \leq 18$ Hence, the solution region is *below* the graph of $x + 3y = 18$.
$2(5) + 1 = 11 \not\geq 16$ Hence, the solution region is *above* the graph of $2x + y = 16$.
$5 \geq 0$
$1 \geq 0$
Thus the solution region is region IV in the diagram.
The corner points are the labeled points (6, 4), (8, 0), and (18, 0).

27. Choose a suitable test point that lies on none of the lines, say (5, 1).
$5 + 3(1) = 8 \not\geq 18$ Hence, the solution region is *above* the graph of $x + 3y = 18$.
$2(5) + 1 = 11 \not\geq 16$ Hence, the solution region is *above* the graph of $2x + y = 16$.
$5 \geq 0$
$1 \geq 0$
Thus the solution region is region I in the diagram.
The corner points are the labeled points (0, 16), (6, 4), and (18, 0).

29. The solution region is bounded (contained in, for example, the circle $x^2 + y^2 = 16$).
The corner points are obvious from the graph:
(0, 0), (0, 2), (3, 0).



31. The solution region is unbounded. The corner points are obvious from the graph: (0, 4) and (5, 0).

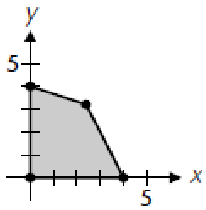


33. The solution region is bounded. Three corner points are

35. The solution region is unbounded. Two corner obvious from the graph: (0, 4), (0, 0), (4, 0). The fourth corner point is obtained by solving the system

$x + 3y = 12$

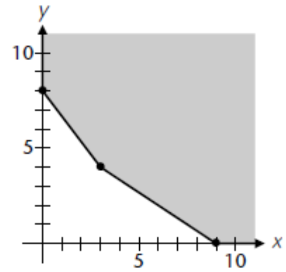
$2x + y = 8$ to obtain $\left(\frac{12}{5}, \frac{16}{5}\right)$.



points are obvious from the graph: (9, 0) and (0, 8). The third corner point is obtained by solving the system

$4x + 3y = 24$

$2x + 3y = 18$ to obtain (3, 4).



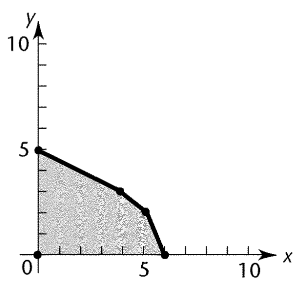
37. The solution region is bounded. Three corner points are obvious from the graph: (6, 0), (0, 0), and (0, 5). The other corner points are obtained by solving:

$2x + y = 12$ and $x + y = 7$

$x + y = 7$ $\quad$ $x + 2y = 10$

to obtain $\quad$ to obtain

(5, 2) $\quad$ (4, 3)



39. The solution region is unbounded. Two of the corner points are obvious from the graph:

(16, 0) and (0, 14).

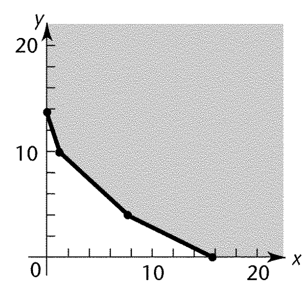
The other corner points are obtained by solving:

$x + 2y = 16$ and $x + y = 12$

$x + y = 12$ $\quad$ $2x + y = 14$

to obtain $\quad$ to obtain

(8, 4) $\quad$ (2, 10)



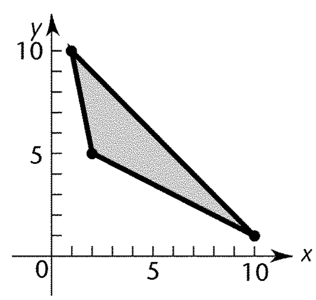
41. The solution region is bounded.

The corner points are obtained by solving:

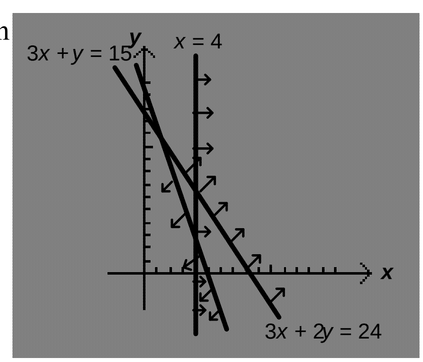
$\begin{cases} x + y = 11 \\ 5x + y = 15 \end{cases}$ to obtain (1, 10)

$\begin{cases} 5x + y = 15 \\ x + 2y = 12 \end{cases}$ to obtain (2, 5), and

$\begin{cases} x + y = 11 \\ x + 2y = 12 \end{cases}$ to obtain (10, 1)



43. From the graph it should be clear that there is no point with x coordinate greater than 4 which satisfies both $3x + 2y \geq 24$ (arrows pointing, roughly, northeast) and $3x + y \leq 15$ (arrows pointing, roughly, southwest). The feasible region is empty.



45. The feasible region is bounded.

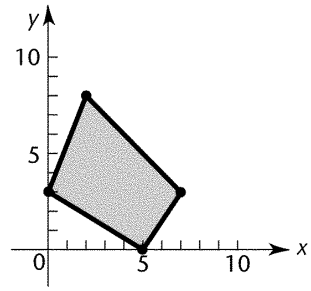
The corner points are obtained by solving:

$\begin{cases} x+y=10 \\ 3x-2y=15 \end{cases}$ to obtain (7, 3)

$\begin{cases} 3x-2y=15 \\ 3x+5y=15 \end{cases}$ to obtain (5, 0),

$\begin{cases} 3x+5y=15 \\ -5x+2y=6 \end{cases}$ to obtain (0, 3), and

$\begin{cases} -5x+2y=6 \\ x+y=10 \end{cases}$ to obtain (2, 8)

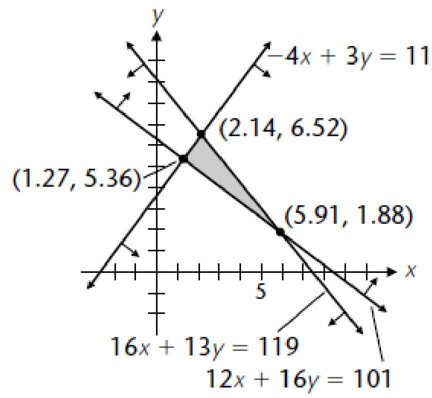


47. The feasible region is bounded.
The corner points are obtained by solving:

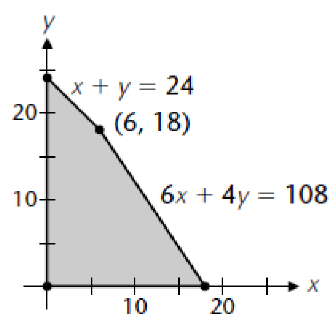
$\begin{cases} 16x+13y=119 \\ 12x+16y=101 \end{cases}$ to obtain (5.91, 1.88)

$\begin{cases} 16x+13y=119 \\ -4x+3y=11 \end{cases}$ to obtain (2.14, 6.53) and

$\begin{cases} 12x+16y=101 \\ -4x+3y=11 \end{cases}$ to obtain (1.27, 5.36)



49. Let x = number of trick skis produced per day.
y = number of slalom skis produced per day
Clearly x and y must be non–negative.
Hence $x \geq 0$ (1)
$y \geq 0$ (2)
To fabricate x trick skis requires $6x$ hours. To fabricate y slalom skis requires $4y$ hours. 108 hours are available for fabricating; hence
$6x + 4y \leq 108$ (3)
To finish x trick skis requires $1x$ hours.
To finish y slalom skis requires $1y$ hours.
24 hours are available for finishing, hence
$x + y \leq 24$ (4)
Graphing the inequality system (1), (2), (3), (4), we have the diagram.



51. (A) All production schedules in the feasible region that are on the graph of $50x + 60y = 1{,}100$ will result in a profit of \$1,100.

(B) There are many possible choices. For example, producing 5 trick and 15 slalom skis will produce a profit of \$1,150. The graph of the line $50x + 60y = 1{,}150$ includes all the production schedules in the feasible region that result in a profit of \$1,150.

(C) A graphical approach would involve drawing other lines of the type $50x + 60y = A$. The graphs of these lines include all production schedules that will result in a profit of A. Increase A until the line either intersects the feasible region only in 1 corner point or contains an edge of the feasible region. This value of A will be the maximum profit possible.

53. Clearly x and y must be non–negative.
Hence $x \geq 0$ (1)

$y \geq 0$ (2)

x cubic yards of mix A contains $20x$ pounds of phosphoric acid.
y cubic yards of mix B contains $10y$ pounds of phosphoric acid.
At least 460 pounds of phosphoric acid are required, hence

$20x + 10y \geq 460$ (3)

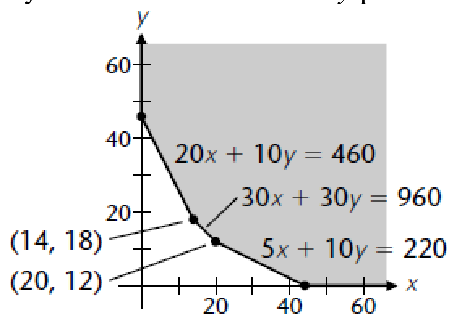
x cubic yards of mix A contains $30x$ pounds of nitrogen.
y cubic yards of mix B contains $30y$ pounds of nitrogen.
At least 960 pounds of nitrogen are required, hence

$30x + 30y \geq 960$ (4)

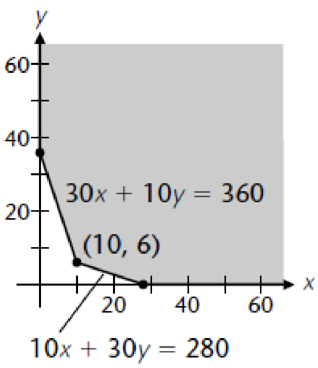
x cubic yards of mix A contains $5x$ pounds of potash. y cubic yards of mix B contains $10y$ pounds of potash.
At least 220 pounds of potash are required, hence

$5x + 10y \geq 220$ (5)

Graphing the inequality system (1), (2), (3), (4), (5), we have the diagram:



55. Clearly x and y must be non–negative.
Hence $x \geq 0$ (1)
$y \geq 0$ (2)
Each sociologist will spend 10 hours collecting data: $10x$ hours.
Each research assistant will spend 30 hours collecting data: $30y$ hours.
At least 280 hours must be spent collecting data; hence
$10x + 30y \geq 280$ (3)
Each sociologist will spend 30 hours analyzing data: $30x$ hours.
Each research assistant will spend 10 hours analyzing data: $10y$ hours.
At least 360 hours must be spent analyzing data; hence
$30x + 10y \geq 360$ (4)
Graphing the inequality system (1), (2), (3), (4), we have the diagram.



Section 10–8

1. The objective function is the quantity to be maximized or minimized.
3. The problem constraints are the limits imposed by reality.
5. The feasible region is the set of possible values of the variables under the constraints.
7. A corner point is a point where two boundaries of the feasible region intersect.

9.

Corner Point (x, y)	Objective Function $z = x + y$	
(0, 12)	12	
(7, 9)	16	Maximum value
(10, 0)	10	
(0, 0)	0	

The maximum value of z on S is 16 at (7, 9).

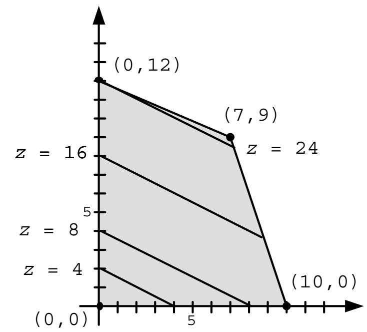
11.

Corner Point	Objective Function	

(x, y)	$z = 3x + 7y$	
(0,12)	84	Maximum value } Multiple optimal solutions
(7,9)	84	Maximum value } Multiple optimal solutions
(10, 0)	30	
(0, 0)	0	

The maximum value of z on S is 84 at both (0, 12) and (7, 9).

13. Plugging in zero for x, we get $z = 2y$ or $y = \frac{z}{2}$. Plugging in zero for y, we get $z = x$. So the intercepts of the line $z = x + 2y$ are $\left(0, \frac{z}{2}\right)$ and $(z, 0)$. The feasible region is shown at the right with several constant z lines drawn in. Sliding a straight edge parallel to these constant z lines in the direction of increasing z, we can see that the point in the feasible region that will intersect the constant z line for largest possible z is (7, 9). When $x = 7$ and $y = 9$, $z = 25$.

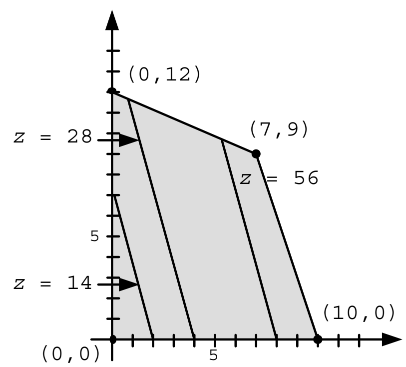


Check:

Corner Point (x, y)	Objective Function $z = x + 2y$	
(0, 12)	24	
(7, 9)	25	Maximum value
(10, 0)	10	
(0, 0)	0	

The maximum value of z on S is 25 at (7, 9).

15. Plugging in zero for x, we get $z = 2y$ or $y = \frac{z}{2}$. Plugging in zero for y, we get $z = 7x$ or $x = \frac{z}{7}$. So the intercepts of the line $z = 7x + 2y$ are $\left(0, \frac{z}{2}\right)$ and $\left(\frac{z}{7}, 0\right)$. The feasible region is shown at the right with several constant z lines drawn in. If we slide a straight edge parallel to the constant z lines as z increases, the last point in S that will intersect our lines is (10, 0). When $x = 10$ and $y = 0$, $z = 70$.



Check:

Corner Point (x, y)	Objective Function $z = 7x + 2y$	
(0, 12)	24	
(7, 9)	67	
(10, 0)	70	Maximum value
(0, 0)	0	

The maximum value of z on S is 70 at (10, 0).

17.

Corner Point (x, y)	Objective Function $z = 7x + 4y$	
(0, 12)	48	
(12, 0)	84	
(4, 3)	40	
(0, 8)	32	Minimum value

The minimum value of z on S is 32 at (0, 8).

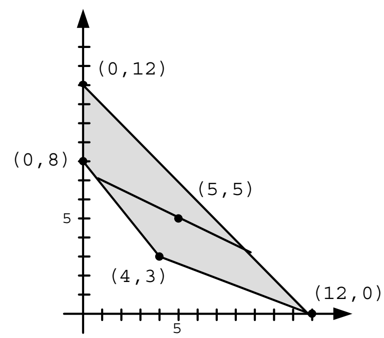
19.

Corner Point (x, y)	Objective Function $3x + 8y$	

(0,12)	96		
(12,0)	36	Minimum value	Multiple optimal solutions
(4,3)	36	Minimum value	
(0,8)	64		

The minimum value of z on S is 36 at both (12, 0) and (4, 3).

21. If $x = 5$ and $y = 5$, $z = 5 + 2(5) = 15$, so the constant–value line we need is $x + 2y = 15$. The feasible region is shown at the right with the constant–value line. If we slide a straightedge parallel to this constant–value line in the direction of decreasing z (downward), the last point in T that will intersect our line is (4, 3). When $x = 4$ and $y = 3$, $z = 10$.

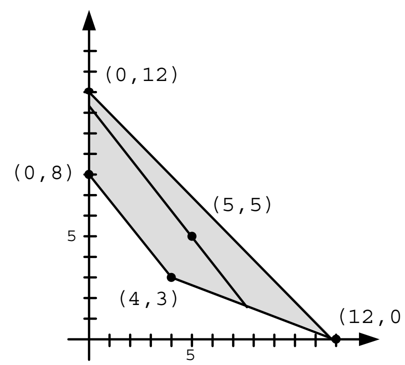


Check:

Corner Point (x, y)	Objective Function $z = x + 2y$	
(0, 12)	24	
(12, 0)	12	
(0, 8)	16	
(4, 3)	10	Minimum value

The minimum value of z on T is 10 at (4, 3).

23. If $x = 5$ and $y = 5$, $z = 5(5) + 4(5) = 45$, so the constant–value line we need is $5x + 4y = 45$. The feasible region is shown at the right with the constant–value line. The constant–value line appears to be parallel to the edge connecting (0, 8) and (4, 3). Thus the minimum could occur at either (0, 8) or (4, 3) when $x = 0$ and $y = 8$, $z = 32$. When $x = 4$ and $y = 3$, $z = 32$ also.



Check:

Corner Point (x, y)	Objective Function $z = 5x + 4y$	
(0, 12)	48	
(12, 0)	60	
(0, 8)	32	Minimum value
(4, 3)	32	Minimum value

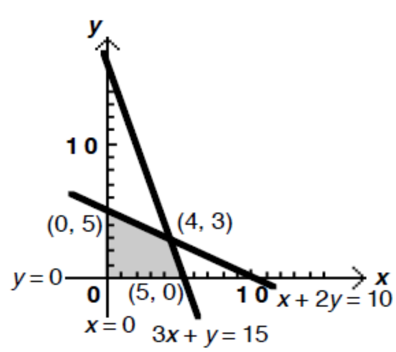
The minimum value of 32 occurs at both (0, 8) and (4, 3).

25. The feasible region is graphed as follows:
The corner points (0, 5), (5, 0) and (0, 0) are obvious from the graph. The corner point (4, 3) is obtained by solving the system

$x + 2y = 10$

$3x + y = 15$

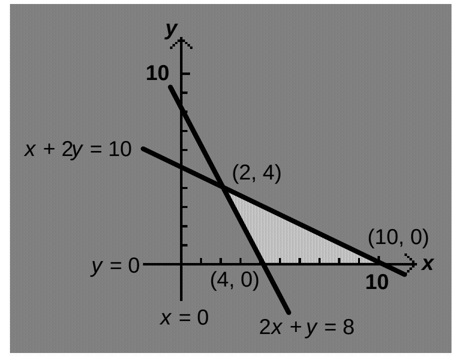
We now evaluate the objective function at each corner point.



Corner Point (x, y)	Objective Function $z = 3x + 2y$	
(0, 5)	10	
(0, 0)	0	
(5, 0)	15	
(4, 3)	18	Maximum value

The maximum value of z on S is 18 at (4, 3).

27. The feasible region is graphed as follows:
The corner points (4, 0) and (10, 0) are obvious from the graph.
The corner point (2, 4) is obtained by solving the system
$x + 2y = 10$
$2x + y = 8$
We now evaluate the objective function at each corner point.



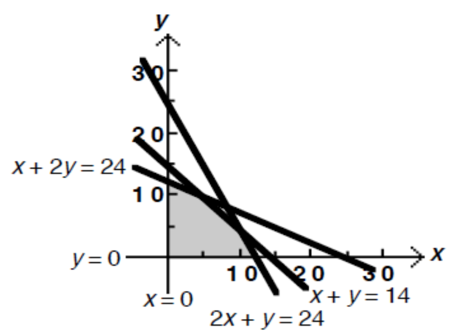
Corner Point (x, y)	Objective Function $z = 3x + 4y$	
(4, 0)	12	Minimum value
(10, 0)	30	
(2, 4)	22	

The minimum value of z on S is 12 at (4, 0).

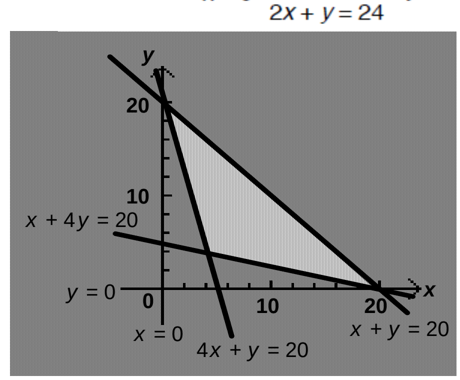
29. The feasible region is graphed below. The corner points (0, 12), (0, 0), and (12, 0) are obvious from the graph. The other corner points are obtained by solving:
$x + 2y = 24$ and $x + y = 14$
$x + y = 14$ to obtain (4, 10) $2x + y = 24$ to obtain (10, 4)
We now evaluate the objective function at each corner point.

Corner Point (x, y)	Objective Function $z = 3x + 4y$	
(0, 12)	48	
(0, 0)	0	
(12, 0)	36	
(10, 4)	46	
(4, 10)	52	Maximum value

The maximum value of z on S is 52 at (4, 10).



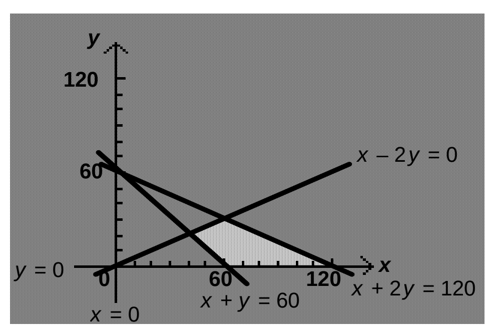
31. The feasible region is graphed as follows:
The corner points (0, 20) and (20, 0) are obvious from the graph.
The third corner point is obtained by solving:
$x + 4y = 20$
$4x + y = 20$ to obtain (4, 4)
We now evaluate the objective function at each corner point.



Corner Point (x, y)	Objective Function $z = 5x + 6y$	
(0, 20)	120	
(20, 0)	100	
(4, 4)	44	Minimum value

The minimum value of z on S is 44 at (4, 4).

33. The feasible region is graphed as follows:
The corner points (60, 0) and (120, 0) are obvious from the graph.
The other corner points are obtained by solving:
$x + y = 60$ and $x + 2y = 120$
$x - 2y = 0$ to obtain (40, 20) $x - 2y = 0$ to obtain (60, 30)
We now evaluate the objective function at each corner point.



Corner Point	Objective Function	

(x, y)	$25x + 50y$	
(60, 0)	1,500	Minimum value
(40, 20)	2,000	
(60, 30)	3,000	Maximum value } Multiple optimal solutions
(120, 0)	3,000	Maximum value } Multiple optimal solutions

The minimum value of z on S is 1,500 at (60, 0).
The maximum value of z on S is 3,000 at (60, 30) and (120, 0).

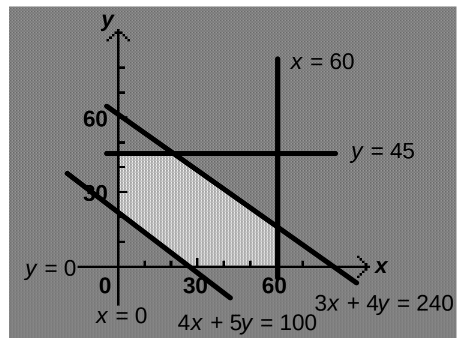
35. The feasible region is graphed as follows:
The corner points (0, 45), (0, 20), (25, 0), and (60, 0) are obvious from the graph shown below.
The other corner points are obtained by solving:

$3x + 4y = 240$ and $3x + 4y = 240$

$y = 45$ to obtain (20, 45) $\quad$ $x = 60$ to obtain (60, 15)

We now evaluate the objective function at each corner point.

Corner Point (x, y)	Objective Function $25x + 15y$	
(0, 45)	675	
(0, 20)	300	Minimum Value
(25, 0)	625	
(60, 0)	1,500	
(60, 15)	1,725	Maximum Value
(20, 45)	1,175	



The minimum value of z on S is 300 at (0, 20).
The maximum value of z on S is 1,725 at (60, 15).

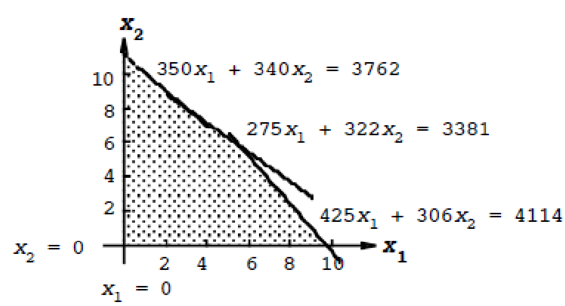
37. The feasible region is graphed as shown at the right.
The corner point (0, 0) is obvious from the graph. The other corner points are obtained by solving:

$x_1 = 0$
$275x_1 + 322x_2 = 3381$
to obtain (0, 10.5)

$350x_1 + 340x_2 = 3762$
$275x_1 + 322x_2 = 3381$
to obtain (3.22, 7.75)

$350x_1 + 340x_2 = 3762$
$425x_1 + 306x_2 = 4114$
to obtain (6.62, 4.25)

$425x_1 + 306x_2 = 4114$
$x_2 = 0$
to obtain (9.68, 0)

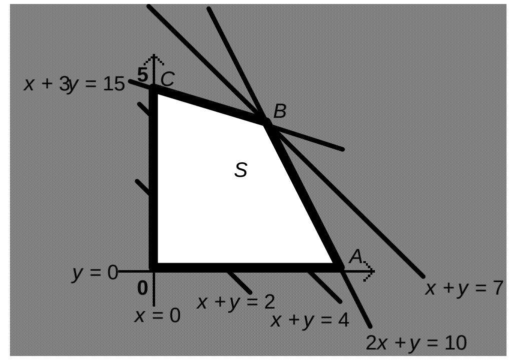


We now evaluate the objective function at each corner point.

Corner Point (x_1, x_2)	Objective Function $525x_1 + 478x_2$	
(0, 0)	0	Minimum value
(0, 10.5)	5019	
(3.22, 7.75)	5395	
(6.62, 4.25)	5507	Maximum value
(9.68, 0)	5082	

The maximum value of P is 5507 at the corner point (6.62, 4.25).

39. The feasible region is graphed as follows: (heavily outlined for clarity) Consider the objective function $x + y$. It should be clear that it takes on the value 2 along $x + y = 2$, the value 4 along $x + y = 4$, the value 7 along $x + y = 7$, and so on. The maximum value of the objective function, then, on S, is 7, which occurs at B. Graphically this occurs when the line $x + y = c$ coincides with the boundary of S. Thus to answer questions (A)-(E) we must determine values of a and b such that the appropriate line $ax + by = c$ coincides with the boundary of S only at the specified points.



(A) The line $ax + by = c$ must have slope negative, but greater in absolute value that that of line segment AB, $2x + y = 10$. Therefore $a > 2b$.

(B) The line $ax + by = c$ must have slope negative but between that of $x + 3y = 15$ and $2x + y = 10$. Therefore $\frac{1}{3}b < a < 2b$.

(C) The line $ax + by = c$ must have slope greater than that of line segment BC, $x + 3y = 15$. Therefore $a < \frac{1}{3}b$ or $b > 3a$.

(D) The line $ax + by = c$ must be parallel to line segment AB, therefore $a = 2b$.

(E) The line $ax + by = c$ must be parallel to line segment BC, therefore $b = 3a$.

41. We let x = the number of trick skis
y = the number of slalom skis

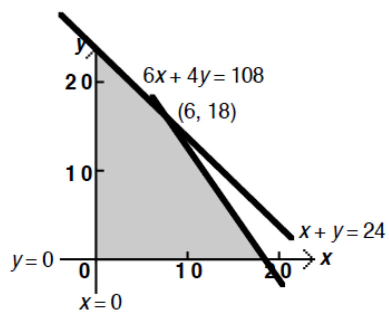
The problem constraints were

$6x + 4y \le 108$
$x + y \le 24$

The non–negative constraints were

$x \ge 0$
$y \ge 0$

The feasible region was graphed there.



(A) We note now: the linear objective function $P = 40x + 30y$ represents the profit.

Three of the corner points are obvious from the graph: (0, 24), (0, 0), and (18, 0). The fourth corner point is obtained by solving: $6x + 4y = 108$
$x + y = 24$ to obtain (6, 18).

Summarizing: the mathematical model for this problem is: Maximize $P = 40x + 30y$
subject to: $6x + 4y \le 108$
$x + y \le 24$
$x, y \ge 0$

We now evaluate the objective function $40x + 30y$ at each corner point.

Corner Point (x, y)	Objective Function $40x + 30y$	
(0, 0)	0	
(18, 0)	720	
(6, 18)	780	Maximum value
(0, 24)	720	

The optimal value is 780 at the corner point (6, 18). Thus, 6 trick skis and 18 slalom skis should be manufactured to obtain the maximum profit of $780.

(B) The objective function now becomes $40x + 25y$. We evaluate this at each corner point.

Corner Point (x, y)	Objective Function $40x + 25y$	
(0, 0)	0	
(18, 0)	720	Maximum value
(6, 18)	690	
(0, 24)	600	

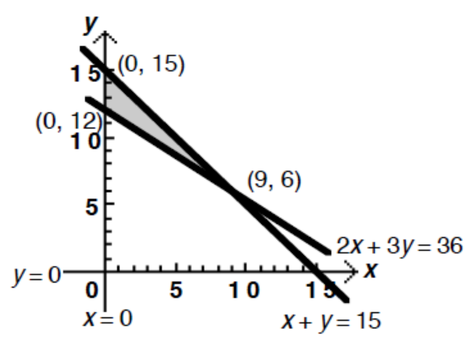
The optimal value is now 720 at the corner point (18, 0). Thus, 18 trick skis and no slalom skis should be produced to obtain a maximum profit of \$720.

(C) The objective function now becomes $40x + 45y$. We evaluate this at each corner point.

Corner Point (x, y)	Objective Function $40x + 45y$	
(0, 0)	0	
(18, 0)	720	
(6, 18)	1,050	
(0, 24)	1,080	Maximum value

The optimal value is now 1,080 at the corner point (0, 24). Thus, no trick skis and 24 slalom skis should be produced to obtain a maximum profit of \$1,080.

43. Let x = number of model A trucks
y = number of model B trucks
We form the linear objective function $C = 15{,}000x + 24{,}000y$
We wish to minimize C, the cost of buying x trucks @ \$15,000 and y trucks @ \$24,000, subject to the constraints.
$x + y \le 15$ maximum number of trucks constraint
$2x + 3y \ge 36$ capacity constraint
$x, y \ge 0$ non–negative constraints.
Solving the system of constraint inequalities graphically, we obtain the feasible region S shown in the diagram.

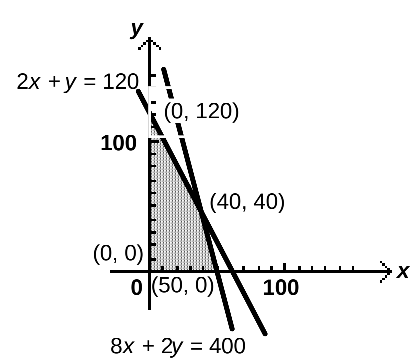


Next we evaluate the objective function at each corner point.

Corner Point (x, y)	Objective Function $C = 15{,}000x + 24{,}000y$	
(0, 12)	288,000	
(0, 15)	360,000	
(9, 6)	279,000	Minimum value

The optimal value is \$279,000 at the corner point (9, 6). Thus, the company should purchase 9 model A trucks and 6 model B trucks to realize the minimum cost of \$279,000.

45. (A) Let x = number of tables
y = number of chairs
We form the linear objective function $P = 90x + 25y$
We wish to maximize P, the profit from x tables @ \$90 and y chairs @ \$25, subject to the constraints
$8x + 2y \le 400$ assembly department constraint
$2x + y \le 120$ finishing department constraint
$x, y \ge 0$ non–negative constraints
Solving the system of constraint inequalities graphically, we obtain the feasible region S' shown in the diagram

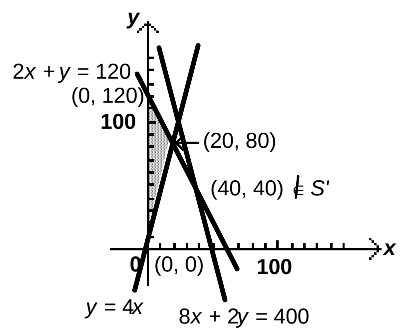


Next we evaluate the objective function at each corner point.

Corner Point (x, y)	Objective Function $P = 90x + 25y$	
(0, 0)	0	
(50, 0)	4,500	
(40, 40)	4,600	Maximum value
(0, 120)	3,000	

The optimal value is 4,600 at the corner point (40, 40). Thus, the company should manufacture 40 tables and 40 chairs for a maximum profit of $4,600.

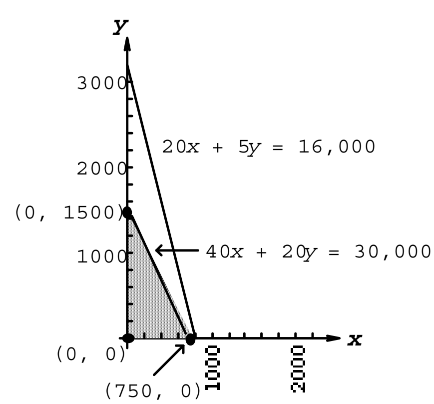
(B) We are faced with the further condition that $y \geq 4x$.
We wish, then, to maximize $P = 90x + 25y$ under the constraints
$8x + 2y \leq 400$
$2x + y \leq 120$
$y \geq 4x$
$x, y \geq 0$
The feasible region is now S' as graphed.
Note that the new condition has the effect of excluding (40, 40) from the feasible region.
We now evaluate the objective function at the new corner points.



Corner Point (x, y)	Objective Function $P = 90x + 25y$	
(0, 120)	3,000	
(0, 0)	0	
(20, 80)	3,800	Maximum value

The optimal value is now 3,800 at the corner point (20, 80). Thus the company should manufacture 20 tables and 80 chairs for a maximum profit of $3,800.

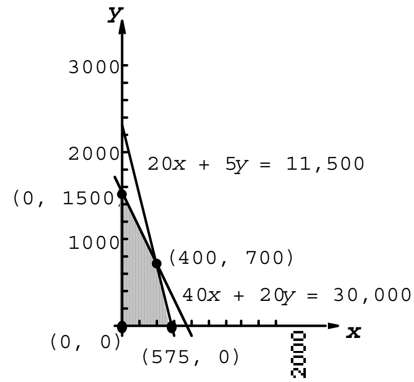
47. Let x = number of gallons produced using the old process
y = number of gallons produced using the new process
We form the linear objective function $P = 0.6x + 0.2y$
(A) We wish to maximize P, the profit from x gallons using the old process and y gallons using the new process, subject to the constraints
$20x + 5y \leq 16{,}000$ sulfur dioxide constraint
$40x + 20y \leq 30{,}000$ particulate matter constraint
$x, y \geq 0$ non–negative constraints
Solving the system of constraint inequalities graphically, we obtain the feasible region S shown in the diagram. Note that no corner points are determined by this (very weak) sulfur dioxide constraint.
We evaluate the objective function at each corner point.



Corner Point (x, y)	Objective Function $P = 0.6x + 0.2y$	
(0, 0)	0	
(0, 1500)	300	
(750, 0)	450	Maximum value

The optimal value is 450 at the corner point (750, 0). Thus, the company should manufacture 750 gallons by the old process exclusively, for a profit of $450.

(B) The sulfur dioxide constraint is now
$20x + 5y \le 11{,}500$.
We now wish to maximize P subject to the constraints
$20x + 5y \le 11{,}500$
$40x + 20y \le 30{,}000$
$x, y \ge 0$
The feasible region is now S_1 as shown.

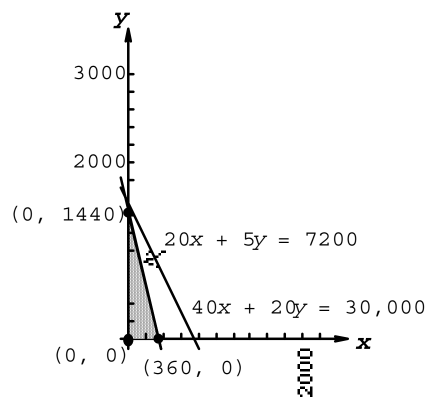


We evaluate the objective function at the new corner points.

Corner Point (x, y)	Objective Function $P = 0.6x + 0.2y$	
(0, 0)	0	
(575, 0)	345	
(400, 700)	380	Maximum value
(0, 1500)	300	

The optimal value is now 380 at the corner point (400, 700). Thus, the company should manufacture 400 gallons by the old process and 700 gallons by the new process, for a profit of \$380.

(C) The sulfur dioxide constraint is now
$20x + 5y \le 7{,}200$.
We now wish to maximize P subject to the constraints
$20x + 5y \le 7{,}200$
$40x + 20y \le 30{,}000$
$x, y \ge 0$
The feasible region is now S_2 as shown.

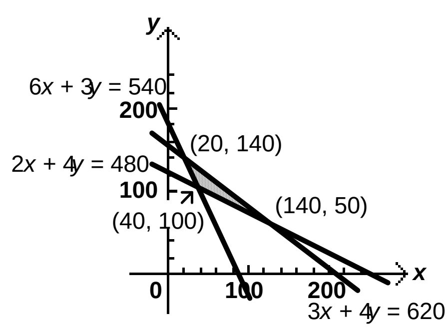


Note that now no corner points are determined by the particulate matter constraint.
We evaluate the objective function at the new corner points.

Corner Point (x, y)	Objective Function $P = 0.6x + 0.2y$	
(0, 0)	0	
(360, 0)	216	
(0, 1440)	288	Maximum value

The optimal value is now 288 at the corner point (0, 1440). Thus, the company should manufacture 1,440 gallons by the new process exclusively, for a profit of \$288.

49. (A) Let x = number of bags of Brand A
y = number of bags of Brand B
We form the objective function $N = 6x + 7y$
N represents the amount of nitrogen in x bags @ 6 pounds per bag and y bags @ 7 pounds per bag.
We wish to optimize N subject to the constraints
$2x + 4y \ge 480$ phosphoric acid constraint
$6x + 3y \ge 540$ potash constraint
$3x + 4y \le 620$ chlorine constraint
$x, y \ge 0$ non-negative constraints
Solving the system of constraint inequalities graphically, we obtain the feasible region S shown in the diagram.
Next we evaluate the objective function at the corner points.



Corner Point (x, y)	Objective Function $N = 6x + 7y$	
$(20, 140)$	1,100	
$(40, 100)$	940	Minimum value
$(140, 50)$	1,190	Maximum value

So the nitrogen will range from a minimum of 940 pounds when 40 bags of brand *A* and 100 bags of Brand *B* are used to a maximum of 1,190 pounds when 140 bags of brand *A* and 50 bags of brand *B* are used.

CHAPTER 10 REVIEW

1. $2x + y = 7$
$3x - 2y = 0$
We multiply the top equation by 2 and add.
$$\begin{aligned} 4x + 2y &= 14 \\ \underline{3x - 2y} &\underline{= 0} \\ 7x &= 14 \\ x &= 2 \end{aligned}$$
Substituting $x = 2$ in the top equation, we have
$2(2) + y = 7$
$y = 3$
Solution: (2, 3) *(10–1)*

2. $3x - 6y = 5$
$-2x + 4y = 1$
We multiply the top equation by 2, the bottom by 3, and add.
$$\begin{aligned} 6x - 12y &= 10 \\ \underline{-6x + 12y} &\underline{= 3} \\ 0 &= 13 \end{aligned}$$
No solution *(10–1)*

3. $4x - 3y = -8$
$-2x + \frac{3}{2}y = 4$
We multiply the bottom equation by 2 and add.
$$\begin{aligned} 4x - 3y &= -8 \\ \underline{-4x + 3y} &\underline{= 8} \\ 0 &= 0 \end{aligned}$$
There are infinitely many solutions. For any real number t, $4t - 3y = -8$, hence,
$-3y = -4t - 8$
$y = \frac{4t + 8}{3}$ Thus, $\left(t, \frac{4t+8}{3}\right)$ is a solution for any real number t. *(10–1)*

4.
$$\begin{aligned} x - 3y + z &= 4 & E_1 \\ -x + 4y - 4z &= 1 & E_2 \\ 2x - y + 5z &= -3 & E_3 \end{aligned}$$
Add E_1 to E_2 to eliminate x.
Also multiply E_1 by –2 and add to E_3 to eliminate x.
$$\begin{aligned} x - 3y + z &= 4 & E_1 \\ \underline{-x + 4y - 4z} &\underline{= 1} & E_2 \\ y - 3z &= 5 & E_4 \\ -2x + 6y - 2z &= -8 & (-2)E_1 \\ \underline{2x - y + 5z} &\underline{= -3} & E_3 \\ 5y + 3z &= -11 & E_5 \end{aligned}$$
Equivalent system:
$$\begin{aligned} x - 3y + z &= 4 & E_1 \\ y - 3z &= 5 & E_4 \\ 5y + 3z &= -11 & E_5 \end{aligned}$$

5.
$$\begin{aligned} 2x + y - z &= 5 & E_1 \\ x - 2y - 2z &= 4 & E_2 \\ 3x + 4y + 3z &= 3 & E_3 \end{aligned}$$
Multiply E_2 by –2 and add to E_1 to eliminate x.
Also multiply E_2 by –3 and add to E_3 to eliminate x.
$$\begin{aligned} 2x + y - z &= 5 & E_1 \\ \underline{-2x + 4y + 4z} &\underline{= -8} & (-2)E_2 \\ 5y + 3z &= -3 & E_4 \\ 3x + 4y + 3z &= 3 & E_3 \\ \underline{-3x + 6y + 6z} &\underline{= -12} & (-3)E_2 \\ 10y + 9z &= -9 & E_5 \end{aligned}$$
Equivalent system:
$$\begin{aligned} x - 2y - 2z &= 4 & E_2 \\ 5y + 3z &= -3 & E_4 \\ 10y + 9z &= -9 & E_5 \end{aligned}$$

Add E_4 to E_5 to eliminate z

$$\begin{array}{rll} y - 3z &= 5 & E_4 \\ 5y + 3z &= -11 & E_5 \\ \hline 6y \quad &= -6 & \\ y &= -1 & \end{array}$$

Substitute $y = -1$ into E_5 and solve for z.

$$\begin{array}{rll} 5y + 3z &= -11 & E_5 \\ 5(-1) + 3z &= -11 & \\ z &= -2 & \end{array}$$

Substitute $y = -1$ and $z = -2$ into E_1 and solve for x.

$$\begin{array}{rll} x - 3y + z &= 4 & E_1 \\ x - 3(-1) + (-2) &= 4 & \\ x &= 3 & \end{array}$$

(3, –1, –2) *(10–1)*

Multiply E_4 by –2 and add to E_5 to eliminate y.

$$\begin{array}{rll} -10y - 6z &= 6 & (-2)E_4 \\ 10y + 9z &= -9 & E_5 \\ \hline 3z &= -3 & \\ z &= -1 & \end{array}$$

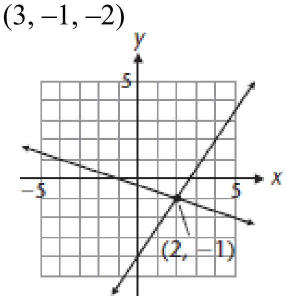
Substitute $z = -1$ into E_4 and solve for y.

$$\begin{array}{rll} 5y + 3z &= -3 & E_4 \\ 5y + 3(-1) &= -3 & \\ y &= 0 & \end{array}$$

Substitute $y = 0$ and $z = -1$ into E_2 and solve for x.

$$\begin{array}{rll} x - 2y - 2z &= 4 & E_2 \\ x - 2(0) - 2(-1) &= 4 & \\ x &= 2 & \end{array}$$

(2, 0, –1) *(10–1)*

6. 

(10–1)

7. $R_1 \leftrightarrow R_2$ means interchange Rows 1 and 2.

$$\left[\begin{array}{rr|r} 3 & -6 & 12 \\ 1 & -4 & 5 \end{array}\right]$$ *(10–2)*

8. $\frac{1}{3}R_2 \to R_2$ means multiply Row 2 by $\frac{1}{3}$.

$$\left[\begin{array}{rr|r} 1 & -4 & 5 \\ 1 & -2 & 4 \end{array}\right]$$ *(10–2)*

9. $(-3)R_1 + R_2 \to R_2$ means replace Row 2 by itself plus –3 times Row 1. $\left[\begin{array}{rr|r} 1 & -4 & 5 \\ 0 & 6 & -3 \end{array}\right]$ *(10–2)*

10. $x_1 = 4$
$x_2 = -7$
The solution is (4, –7) *(10–2)*

11. $x_1 - x_2 = 4$
$0 = 1$
No solution *(10–2)*

12. $x_1 - x_2 = 4$
$0 = 0$
Solution: $x_2 = t$
$x_1 = x_2 + 4 = t + 4$
Thus $x_1 = t + 4$, $x_2 = t$ is the solution, for t any real number. *(10–2)*

13. $AB = \begin{bmatrix} 4 & -2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 5 \\ -4 & 6 \end{bmatrix} = \begin{bmatrix} 4(-1)+(-2)(-4) & 4\cdot 5+(-2)6 \\ 0(-1)+3(-4) & 0\cdot 5+3\cdot 6 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ -12 & 18 \end{bmatrix}$ *(10–3)*

14. $CD = [-1 \quad 4]\begin{bmatrix} 3 \\ -2 \end{bmatrix} = [(-1)3 + 4(-2)] = [-11]$ *(10–3)*

15. $CB = [-1 \quad 4]\begin{bmatrix} -1 & 5 \\ -4 & 6 \end{bmatrix} = [(-1)(-1) + 4(-4) \quad (-1)5 + 4 \cdot 6] = [-15 \quad 19]$ *(10–3)*

16. $AD = \begin{bmatrix} 4 & -2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 4\cdot 3+(-2)(-2) \\ 0\cdot 3+3(-2) \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \end{bmatrix}$ *(10–3)*

17. $A + B = \begin{bmatrix} 4 & -2 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} -1 & 5 \\ -4 & 6 \end{bmatrix} = \begin{bmatrix} 4+(-1) & (-2)+5 \\ 0+(-4) & 3+6 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ -4 & 9 \end{bmatrix}$ (*10–3*)

18. $C + D$ is not defined (*10–3*)

19. $A + C$ is not defined (*10–3*)

20. $2A - 5B = 2\begin{bmatrix} 4 & -2 \\ 0 & 3 \end{bmatrix} - 5\begin{bmatrix} -1 & 5 \\ -4 & 6 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} -5 & 25 \\ -20 & 30 \end{bmatrix} = \begin{bmatrix} 13 & -29 \\ 20 & -24 \end{bmatrix}$ (*10–3*)

21. $CA + C = [-1\ \ 4]\begin{bmatrix} 4 & -2 \\ 0 & 3 \end{bmatrix} + [-1\ \ 4] = [(-1)4 + 4 \cdot 0 \quad (-1)(-2) + 4 \cdot 3] + [-1\ \ 4]$

$= [-4\ \ 14] + [-1\ \ 4] = [-5\ \ 18]$ (*10–3*)

22. $\left[\begin{array}{cc|cc} 4 & 7 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{array}\right] R_1 \leftrightarrow R_2 \sim \left[\begin{array}{cc|cc} -1 & -2 & 0 & 1 \\ 4 & 7 & 1 & 0 \end{array}\right] 4R_1 + R_2 \to R_2 \sim \left[\begin{array}{cc|cc} -1 & -2 & 0 & 1 \\ 0 & -1 & 1 & 4 \end{array}\right] (-2)R_2 + R_1 \to R_1$

$\sim \left[\begin{array}{cc|cc} -1 & 0 & -2 & -7 \\ 0 & -1 & 1 & 4 \end{array}\right] \begin{array}{l} (-1)R_1 \to R_1 \\ (-1)R_2 \to R_2 \end{array} \sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & 7 \\ 0 & 1 & -1 & -4 \end{array}\right]$

Hence, $A^{-1} = \begin{bmatrix} 2 & 7 \\ -1 & -4 \end{bmatrix}$

$A^{-1}A = \begin{bmatrix} 2 & 7 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} 4 & 7 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 2\cdot4+7(-1) & 2\cdot7+7(-2) \\ (-1)4+(-4)(-1) & (-1)7+(-4)(-2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

(*10–4*)

23. As a matrix equation the system becomes

$$\underset{A}{\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}} \underset{X}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{B}{\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}}$$

The solution of $AX = B$ is $X = A^{-1}B$.

Applying matrix methods, we obtain $A^{-1} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$ (details omitted). Applying this inverse, we have

$X = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$

(A) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3\cdot3+(-2)5 \\ (-4)3+3\cdot5 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ $x_1 = -1, x_2 = 3$

(B) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 10 \end{bmatrix} = \begin{bmatrix} 3\cdot7+(-2)10 \\ (-4)7+3\cdot10 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $x_1 = 1, x_2 = 2$

(C) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3\cdot4+(-2)2 \\ (-4)4+3\cdot2 \end{bmatrix} = \begin{bmatrix} 8 \\ -10 \end{bmatrix}$ $x_1 = 8, x_2 = -10$ (*10–4*)

24. $\begin{vmatrix} 2 & -3 \\ -5 & -1 \end{vmatrix} = 2(-1) - (-5)(-3) = -17$ (*10–5*)

25. $\begin{vmatrix} 2 & 3 & -4 \\ 0 & 5 & 0 \\ 1 & -4 & -2 \end{vmatrix} = 0 + 5(-1)^{2+2}\begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} + 0 = 5(-1)^4[2(-2) - 1(-4)] = 0$ (*10–5*)

26. $D = \begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} = 11 \qquad x = \dfrac{\begin{vmatrix} 8 & -2 \\ -1 & 3 \end{vmatrix}}{D} = \dfrac{22}{11} = 2 \quad y = \dfrac{\begin{vmatrix} 3 & 8 \\ 1 & -1 \end{vmatrix}}{D} = \dfrac{-11}{11} = -1$ (*10–5*)

27. We write the augmented matrix:

$\left[\begin{array}{rr|r} 1 & -1 & 4 \\ 2 & 1 & 2 \end{array}\right] \quad (-2)R_1 + R_2 \to R_2$

$-2 \quad 2 \quad -8$

Need a 0 here

$\sim \left[\begin{array}{rr|r} 1 & -1 & 4 \\ 0 & 3 & -6 \end{array}\right] \quad \tfrac{1}{3}R_2 \to R_2$ corresponds to the linear system $\begin{cases} x_1 - x_2 = 4 \\ 3x_2 = -6 \end{cases}$

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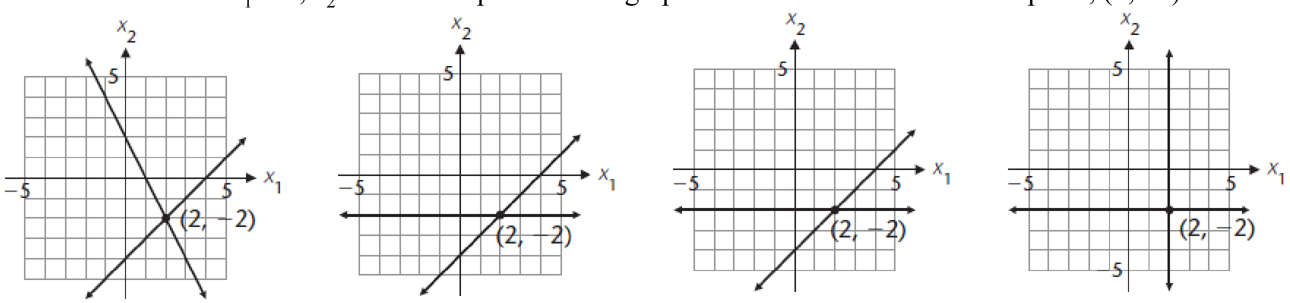
$\sim \left[\begin{array}{rr|r} 1 & -1 & 4 \\ 0 & 1 & -2 \end{array}\right] \quad R_2 + R_1 \to R_1$ corresponds to the linear system $\begin{cases} x_1 - x_2 = 4 \\ x_2 = -2 \end{cases}$

$\sim \left[\begin{array}{rr|r} 1 & 0 & 2 \\ 0 & 1 & -2 \end{array}\right]$ corresponds to the linear system $\begin{cases} x_1 = 2 \\ x_2 = -2 \end{cases}$

The solution is $x_1 = 2$, $x_2 = -2$. Each pair of lines graphed has the same intersection point, (2, –2).



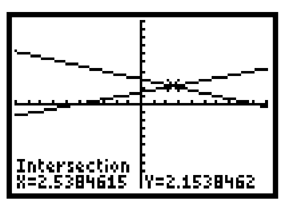
$x_1 - x_2 = 4$, $2x_1 + x_2 = 2$ | $x_1 - x_2 = 4$, $3x_2 = -6$ | $x_1 - x_2 = 4$, $x_2 = -2$ | $x_1 = 2$, $x_2 = -2$ (*10–1, 10–2*)

28. Before we can enter these equations in our graphing calculator, we must solve for y:

$x + 3y = 9 \qquad\qquad -2x + 7y = 10$

$3y = 9 - x \qquad\qquad 7y = 10 + 2x$

$y = \dfrac{9 - x}{3} \qquad\qquad y = \dfrac{10 + 2x}{7}$



Entering these equations into a graphing calculator and applying an intersection routine yields the solution (2.54, 2.15) to two decimal places. (*10–1*)

29. $\left[\begin{array}{cc|c} 3 & 2 & 3 \\ 1 & 3 & 8 \end{array}\right] R_1 \leftrightarrow R_2 \sim \left[\begin{array}{cc|c} 1 & 3 & 8 \\ 3 & 2 & 3 \end{array}\right] (-3)R_1 + R_2 \to R_2 \sim \left[\begin{array}{cc|c} 1 & 3 & 8 \\ 0 & -7 & -21 \end{array}\right] \left(-\frac{1}{7}\right) R_2 \to R_2 \sim \left[\begin{array}{cc|c} 1 & 3 & 8 \\ 0 & 1 & 3 \end{array}\right] (-3)R_2 + R_1 \to R_1$

$-3\ -9\ -24$ $\qquad$ $0\ -3\ -9$

$\sim \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 3 \end{array}\right]$ Solution: $x_1 = -1,\ x_2 = 3$ (*10–2*)

30. $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \end{array}\right] (-1)R_1 + R_2 \to R_2$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 2 & 4 \end{array}\right] (-1)R_2 \to R_2$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 2 & 4 \end{array}\right] \begin{array}{l} (-1)R_2 + R_1 \to R_1 \\ (-1)R_2 + R_3 \to R_3 \\ \end{array}$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_3 + R_1 \to R_1 \\ (-1)R_3 + R_2 \to R_2 \\ \end{array}$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$

Solution: $x_1 = -1,\ x_2 = 2,\ x_3 = 1$ (*10–2*)

31. $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 1 & 3 \end{array}\right] \begin{array}{l} \\ (-2)R_1 + R_2 \to R_2 \\ (-1)R_1 + R_3 \to R_3 \end{array}$

$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -2 & 2 \end{array}\right] \begin{array}{l} \\ (-1)R_2 \to R_2 \\ -\frac{1}{2}R_3 \to R_3 \end{array}$

$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right] (-2)R_2 + R_1 \to R_1$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} R_3 + R_1 \to R_1 \\ (-2)R_3 + R_2 \to R_2 \\ \end{array}$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$

Solution: $x_1 = 2,\ x_2 = 1,\ x_3 = -1$ (*10–2*)

32. $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & 3 & 1 & -3 \\ 3 & 5 & 0 & -1 \end{array}\right] \begin{array}{l} \\ (-2)R_1 + R_2 \to R_2 \\ (-3)R_1 + R_3 \to R_3 \end{array}$

$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -1 & 3 & -7 \\ 0 & -1 & 3 & -7 \end{array}\right] (-1)R_2 + R_3 \to R_3$

$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -1 & 3 & -7 \\ 0 & 0 & 0 & 0 \end{array}\right] 2R_2 + R_1 \to R_1$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & -12 \\ 0 & -1 & 3 & -7 \\ 0 & 0 & 0 & 0 \end{array}\right] (-1)R_2 \to R_2$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & -12 \\ 0 & 1 & -3 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right]$

This corresponds to the system

$x_1 + \quad 5x_3 = -12$

$x_2 - 3x_3 = 7$

Let $x_3 = t$

Then $x_2 = 3x_3 + 7 = 3t + 7$

$x_1 = -5x_3 - 12 = -5t - 12$

Hence $x_1 = -5t - 12$, $x_2 = 3t + 7$, $x_3 = t$ is a solution for every real number t. There are infinitely many solutions.

(*10–2*)

33. $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 2 & -1 & 0 \\ 1 & -3 & -2 \end{array}\right] \begin{array}{l} \\ (-2)R_1 + R_2 \to R_2 \\ (-1)R_1 + R_3 \to R_3 \end{array} \sim \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & -2 \\ 0 & -1 & -3 \end{array}\right] 3R_3 + R_2 \to R_2 \sim \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & -11 \\ 0 & -1 & -3 \end{array}\right]$

The second row corresponds to the equation $0x_1 + 0x_2 = -11$, hence there is no solution. (*10–2*)

34. $\left[\begin{array}{rrr|r} 1 & 2 & -1 & 2 \\ 3 & -1 & 2 & -3 \end{array}\right]$ $(-3)R_1 + R_2 \to R_2$

$\sim \left[\begin{array}{rrr|r} 1 & 2 & -1 & 2 \\ 0 & -7 & 5 & -9 \end{array}\right]$ $-\frac{1}{7}R_2 \to R_2$

$\sim \left[\begin{array}{rrr|r} 1 & 2 & -1 & 2 \\ 0 & 1 & -\frac{5}{7} & \frac{9}{7} \end{array}\right]$ $(-2)R_2 + R_1 \to R_1$

$\sim \left[\begin{array}{rrr|r} 1 & 0 & \frac{3}{7} & -\frac{4}{7} \\ 0 & 1 & -\frac{5}{7} & \frac{9}{7} \end{array}\right]$

This corresponds to the system

$x_1 + \quad \frac{3}{7}x_3 = -\frac{4}{7}$

$x_2 - \frac{5}{7}x_3 = \frac{9}{7}$

Let $x_3 = t$

Then $x_2 = \frac{5}{7}x_3 + \frac{9}{7} = \frac{5}{7}t + \frac{9}{7}$

$x_1 = -\frac{3}{7}x_3 - \frac{4}{7} = -\frac{3}{7}t - \frac{4}{7}$

Hence $x_1 = -\frac{3}{7}t - \frac{4}{7}$, $x_2 = \frac{5}{7}t + \frac{9}{7}$, $x_3 = t$ is a solution for every real number t. There are infinitely many solutions.

(*10–2*)

35. $AD = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} 7 & 0 & -5 \\ 0 & 8 & -2 \end{bmatrix} = \begin{bmatrix} 1\cdot 7+2\cdot 0 & 1\cdot 0+2\cdot 8 & 1(-5)+2(-2) \\ 4\cdot 7+5\cdot 0 & 4\cdot 0+5\cdot 8 & 4(-5)+5(-2) \\ (-3)7+(-1)0 & (-3)0+(-1)8 & (-3)(-5)+(-1)(-2) \end{bmatrix} = \begin{bmatrix} 7 & 16 & -9 \\ 28 & 40 & -30 \\ -21 & -8 & 17 \end{bmatrix}$ (*10–3*)

36. $DA = \begin{bmatrix} 7 & 0 & -5 \\ 0 & 8 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 7\cdot 1+0\cdot 4+(-5)(-3) & 7\cdot 2+0\cdot 5+(-5)(-1) \\ 0\cdot 1+8\cdot 4+(-2)(-3) & 0\cdot 2+8\cdot 5+(-2)(-1) \end{bmatrix} = \begin{bmatrix} 22 & 19 \\ 38 & 42 \end{bmatrix}$ (*10–3*)

37. $BC = \begin{bmatrix} 6 \\ 0 \\ -4 \end{bmatrix}\begin{bmatrix} 2 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 6\cdot 2 & 6\cdot 4 & 6(-1) \\ 0\cdot 2 & 0\cdot 4 & 0(-1) \\ (-4)2 & (-4)4 & (-4)(-1) \end{bmatrix} = \begin{bmatrix} 12 & 24 & -6 \\ 0 & 0 & 0 \\ -8 & -16 & 4 \end{bmatrix}$ (*10–3*)

38. $CB = \begin{bmatrix} 2 & 4 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 0 \\ -4 \end{bmatrix} = [2\cdot 6 + 4\cdot 0 + (-1)(-4)] = [16]$ (*10–3*)

39. Since D has 3 columns and E has 2 rows, DE is not defined. (*10–3*)

40. $ED = \begin{bmatrix} 9 & -3 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} 7 & 0 & -5 \\ 0 & 8 & -2 \end{bmatrix} = \begin{bmatrix} 9\cdot 7+(-3)0 & 9\cdot 0+(-3)8 & 9(-5)+(-3)(-2) \\ (-6)7+2\cdot 0 & (-6)0+2\cdot 8 & (-6)(-5)+2(-2) \end{bmatrix} = \begin{bmatrix} 63 & -24 & -39 \\ -42 & 16 & 26 \end{bmatrix}$ (*10–3*)

41. $\left[\begin{array}{rrr|rrr} 1 & 0 & 4 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -1 & 4 & 0 & 0 & 1 \end{array}\right]$ $\begin{array}{l} 2R_1 + R_2 \to R_2 \\ (-4)R_1 + R_3 \to R_3 \end{array}$ $\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 8 & 2 & 1 & 0 \\ 0 & -1 & -12 & -4 & 0 & 1 \end{array}\right]$ $R_2 + R_3 \to R_3$

$\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 8 & 2 & 1 & 0 \\ 0 & 0 & -4 & -2 & 1 & 1 \end{array}\right]$ $\begin{array}{l} R_3 + R_1 \to R_1 \\ 2R_3 + R_2 \to R_2 \end{array}$ $\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & -2 & 3 & 2 \\ 0 & 0 & -4 & -2 & 1 & 1 \end{array}\right]$ $(-\frac{1}{4})R_3 \to R_3$

$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & -2 & 3 & 2 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$ Hence $A^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ -2 & 3 & 2 \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$

$$A^{-1}A=\begin{bmatrix}-1 & 1 & 1\\ -2 & 3 & 2\\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4}\end{bmatrix}\begin{bmatrix}1 & 0 & 4\\ -2 & 1 & 0\\ 4 & -1 & 4\end{bmatrix}=\begin{bmatrix}(-1)1+1(-2)+1\cdot4 & (-1)0+1\cdot1+1(-1) & (-1)4+1\cdot0+1\cdot4\\ (-2)1+3(-2)+2\cdot4 & (-2)0+3\cdot1+2(-1) & (-2)4+3\cdot0+2\cdot4\\ \left(\frac{1}{2}\right)1+\left(-\frac{1}{4}\right)(-2)+\left(-\frac{1}{4}\right)4 & \left(\frac{1}{2}\right)0+\left(-\frac{1}{4}\right)1+\left(-\frac{1}{4}\right)(-1) & \left(\frac{1}{2}\right)4+\left(-\frac{1}{4}\right)0+\left(-\frac{1}{4}\right)4\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad (10\text{–}4)$$

42. $$\overset{A}{\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 4\\ 1 & 2 & 1\end{bmatrix}}\overset{X}{\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}}=\overset{B}{\begin{bmatrix}k_1\\ k_2\\ k_3\end{bmatrix}}$$

The solution to $AX = B$ is $X = A^{-1}B$.

Applying matrix methods, we obtain $A^{-1}=\begin{bmatrix}-\frac{5}{2} & 2 & -\frac{1}{2}\\ 1 & -1 & 1\\ \frac{1}{2} & 0 & -\frac{1}{2}\end{bmatrix}$ (details omitted).

Applying the inverse, we have

(A) $B=\begin{bmatrix}1\\ 3\\ 3\end{bmatrix}$ $X=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}-\frac{5}{2} & 2 & -\frac{1}{2}\\ 1 & -1 & 1\\ \frac{1}{2} & 0 & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}1\\ 3\\ 3\end{bmatrix}=\begin{bmatrix}2\\ 1\\ -1\end{bmatrix}$ $x_1 = 2,\ x_2 = 1,\ x_3 = -1$

(B) $B=\begin{bmatrix}0\\ 0\\ -2\end{bmatrix}$ $X=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}-\frac{5}{2} & 2 & -\frac{1}{2}\\ 1 & -1 & 1\\ \frac{1}{2} & 0 & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}0\\ 0\\ -2\end{bmatrix}=\begin{bmatrix}1\\ -2\\ 1\end{bmatrix}$ $x_1 = 1,\ x_2 = -2,\ x_3 = 1$

(C) $B=\begin{bmatrix}-3\\ -4\\ 1\end{bmatrix}$ $X=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}-\frac{5}{2} & 2 & -\frac{1}{2}\\ 1 & -1 & 1\\ \frac{1}{2} & 0 & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}-3\\ -4\\ 1\end{bmatrix}=\begin{bmatrix}-1\\ 2\\ -2\end{bmatrix}$ $x_1 = -1, x_2 = 2, x_3 = -2$ $(10\text{–}4)$

43. $$\begin{vmatrix}-\frac{1}{4} & \frac{3}{2}\\ \frac{1}{2} & \frac{2}{3}\end{vmatrix}=\left(-\frac{1}{4}\right)\left(\frac{2}{3}\right)-\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)=-\frac{1}{6}-\frac{3}{4}=-\frac{11}{12} \qquad (10\text{–}5)$$

44. $$\begin{vmatrix}2 & -1 & 1\\ -3 & 5 & 2\\ 1 & -2 & 4\end{vmatrix}=\begin{vmatrix}0 & 0 & 1\\ -7 & 7 & 2\\ -7 & 2 & 4\end{vmatrix}\ \begin{matrix}(-2)C_3+C_1\to C_1\\ C_3+C_2\to C_2\end{matrix}=0+0+1(-1)^{1+3}\begin{vmatrix}-7 & 7\\ -7 & 2\end{vmatrix}=(-1)^4[(-7)2-(-7)7]=35 \qquad (10\text{–}5)$$

45. $$y=\frac{\begin{vmatrix}1 & -6 & 1\\ 0 & 4 & -1\\ 2 & 2 & 1\end{vmatrix}}{\begin{vmatrix}1 & -2 & 1\\ 0 & 1 & -1\\ 2 & 2 & 1\end{vmatrix}}=\frac{\begin{vmatrix}1 & -6 & 1\\ 0 & 4 & -1\\ 0 & 14 & -1\end{vmatrix}}{\begin{vmatrix}1 & -2 & -1\\ 0 & 1 & 0\\ 2 & 2 & 3\end{vmatrix}}=\frac{1(-1)^{1+1}\begin{vmatrix}4 & -1\\ 14 & -1\end{vmatrix}}{1(-1)^{2+2}\begin{vmatrix}1 & -1\\ 2 & 3\end{vmatrix}}=\frac{(-1)^2[4(-1)-14(-1)]}{(-1)^4[1\cdot3-2(-1)]}=\frac{10}{5}=2 \qquad (10\text{–}5)$$

46. $$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7000 \\ 0.04 & 0.05 & 0.06 & 360 \\ 0.04 & 0.05 & -0.06 & 120 \end{array}\right] \begin{array}{l} \\ (-0.04)R_1 + R_2 \to R_2 \\ (-0.04)R_1 + R_3 \to R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 7000 \\ 0 & 0.01 & 0.02 & 80 \\ 0 & 0.01 & -0.1 & -160 \end{array}\right] 100R_2 \to R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 7000 \\ 0 & 1 & 2 & 8000 \\ 0 & 0.01 & -0.1 & -160 \end{array}\right] \begin{array}{l} (-1)R_2 + R_1 \to R_1 \\ \\ (-0.01)R_2 + R_3 \to R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1000 \\ 0 & 1 & 2 & 8000 \\ 0 & 0 & -0.12 & -240 \end{array}\right] -\tfrac{25}{3}R_3 \to R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1000 \\ 0 & 1 & 2 & 8000 \\ 0 & 0 & 1 & 2000 \end{array}\right] \begin{array}{l} R_3 + R_1 \to R_1 \\ (-2)R_3 + R_2 \to R_2 \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1000 \\ 0 & 1 & 0 & 4000 \\ 0 & 0 & 1 & 2000 \end{array}\right]$$

Solution: $x_1 = 1000$, $x_2 = 4000$, $x_3 = 2000$ *(10–2)*

47. $\begin{vmatrix} u+kv & v \\ w+kx & x \end{vmatrix} = (u + kv)x - (w + kx)v = ux + kvx - wv - kvx = ux - wv = \begin{vmatrix} u & v \\ w & x \end{vmatrix}$ *(10–5)*

48.(A) The system is independent. There is one solution.

(B) The matrix is in fact

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & n \end{array}\right]$$

The third row corresponds to the equation $0x_1 + 0x_2 + 0x_3 = n$. This is impossible. The system has no solution.

(C) The matrix is in fact

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, there are an infinite number of solutions ($x_3 = t$, $x_2 = 5 - 2t$, $x_1 = 4 + 3t$, for t any real number.) *(10–2)*

49. (A) If the coefficient matrix has an inverse, then the system can be written as $AX = B$ and its solution can be written $X = A^{-1}B$. Thus the system has one solution.

(B) If the coefficient matrix does not have an inverse, then the system can be solved by Gauss–Jordan elimination, but it will not have exactly one solution. The other possibilities are that the system has no solution or an infinite number of solutions, and either possibility may occur. *(10–4)*

50. If we assume that A is a non–zero matrix with an inverse A^{-1}, then if $A^2 = 0$ we can write $A^{-1}A^2 = A^{-1}0$ or $A^{-1}AA = 0$ or $IA = 0$ or $A = 0$. But A was assumed non–zero, so there is a contradiction. Hence A^{-1} cannot exist for such a matrix. *(10–4)*

51.

$AX - B$	$= CX$	
$AX - B + B - CX$	$= CX - CX + B$	Addition property
$AX + 0 - CX$	$= 0 + B$	$M + (-M) = 0$
$AX - CX$	$= B$	$M + 0 = M$
$(A - C)X$	$= B$	Right distributive property
$(A - C)^{-1}[(A - C)X]$	$= (A - C)^{-1}B$	Left multiplication property
$[(A - C)^{-1}(A - C)]X$	$= (A - C)^{-1}B$	Associative property
IX	$= (A - C)^{-1}B$	$A^{-1}A = I$
X	$= (A - C)^{-1}B$	$IX = X$

Common Errors: $(A - C)^{-1}B \neq B(A - C)^{-1}$
$(A - C)^{-1} \neq A^{-1} - C^{-1}$

(10–4)

52. $$\left[\begin{array}{ccc|ccc} 4 & 5 & 6 & 1 & 0 & 0 \\ 4 & 5 & -6 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] (-1)R_1 + R_2 \to R_2 \sim \left[\begin{array}{ccc|ccc} 4 & 5 & 6 & 1 & 0 & 0 \\ 0 & 0 & -12 & -1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] R_1 \leftrightarrow R_3$$

$$\sim\left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -12 & -1 & 1 & 0 \\ 4 & 5 & 6 & 1 & 0 & 0 \end{array}\right] (-4)R_1 + R_3 \to R_3 \quad \sim\left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -12 & -1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & -4 \end{array}\right] R_2 \leftrightarrow R_3$$

$$\sim\left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 & -4 \\ 0 & 0 & -12 & -1 & 1 & 0 \end{array}\right] (-1)R_2 + R_1 \to R_1 \quad \sim\left[\begin{array}{rrr|rrr} 1 & 0 & -1 & -1 & 0 & 5 \\ 0 & 1 & 2 & 1 & 0 & -4 \\ 0 & 0 & -12 & -1 & 1 & 0 \end{array}\right] -\tfrac{1}{12}R_3 \to R_3$$

$$\sim\left[\begin{array}{rrr|rrr} 1 & 0 & -1 & -1 & 0 & 5 \\ 0 & 1 & 2 & 1 & 0 & -4 \\ 0 & 0 & 1 & \tfrac{1}{12} & -\tfrac{1}{12} & 0 \end{array}\right] \begin{array}{l} R_3 + R_1 \to R_1 \\ (-2)R_3 + R_2 \to R_2 \end{array} \sim\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -\tfrac{11}{12} & -\tfrac{1}{12} & 5 \\ 0 & 1 & 0 & \tfrac{10}{12} & \tfrac{2}{12} & -4 \\ 0 & 0 & 1 & \tfrac{1}{12} & -\tfrac{1}{12} & 0 \end{array}\right]$$

Hence $A^{-1} = \begin{bmatrix} -\tfrac{11}{12} & -\tfrac{1}{12} & 5 \\ \tfrac{10}{12} & \tfrac{2}{12} & -4 \\ \tfrac{1}{12} & -\tfrac{1}{12} & 0 \end{bmatrix}$ or $\tfrac{1}{12}\begin{bmatrix} -11 & -1 & 60 \\ 10 & 2 & -48 \\ 1 & -1 & 0 \end{bmatrix}$

$$A^{-1}A = \tfrac{1}{12}\begin{bmatrix} -11 & -1 & 60 \\ 10 & 2 & -48 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 4 & 5 & 6 \\ 4 & 5 & -6 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \tfrac{1}{12}\begin{bmatrix} (-11)4+(-1)4+60\cdot 1 & (-11)5+(-1)5+60\cdot 1 & (-11)6+(-1)(-6)+60\cdot 1 \\ 10\cdot 4+2\cdot 4+(-48)1 & 10\cdot 5+2\cdot 5+(-48)1 & 10\cdot 6+2(-6)+(-48)1 \\ 1\cdot 4+(-1)4+0\cdot 1 & 1\cdot 5+(-1)5+0\cdot 1 & 1\cdot 6+(-1)(-6)+0\cdot 1 \end{bmatrix}$$

$$= \tfrac{1}{12}\begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \qquad (10\text{–}4)$$

53. Multiplying the first two equations by 100, the system becomes

$4x_1 + 5x_2 + 6x_3 = 36{,}000$

$4x_1 + 5x_2 - 6x_3 = 12{,}000$

$x_1 + x_2 + x_3 = 7{,}000$

As a matrix equation, we have

$$\overset{A}{\begin{bmatrix} 4 & 5 & 6 \\ 4 & 5 & -6 \\ 1 & 1 & 1 \end{bmatrix}}\overset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \overset{B}{\begin{bmatrix} 36{,}000 \\ 12{,}000 \\ 7{,}000 \end{bmatrix}}$$

The solution to $AX = B$ is $X = A^{-1}B$. Using A^{-1} from problem 52, we have

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \tfrac{1}{12}\begin{bmatrix} -11 & -1 & 60 \\ 10 & 2 & -48 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 36{,}000 \\ 12{,}000 \\ 7{,}000 \end{bmatrix} = \tfrac{1}{12}\begin{bmatrix} (-11)(36{,}000)+(-1)(12{,}000)+(60)(7{,}000) \\ (10)(36{,}000)+(2)(12{,}000)+(-48)(7{,}000) \\ 1(36{,}000)+(-1)(12{,}000)+(0)(7{,}000) \end{bmatrix}$$

$$= \tfrac{1}{12}\begin{bmatrix} 12{,}000 \\ 48{,}000 \\ 24{,}000 \end{bmatrix} = \begin{bmatrix} 1{,}000 \\ 4{,}000 \\ 2{,}000 \end{bmatrix}$$

Hence, $x_1 = 1{,}000$, $x_2 = 4{,}000$, $x_3 = 2{,}000$ (*10–4*)

54. Let x = number of $\frac{1}{2}$ – pound packages

y = number of $\frac{1}{3}$ – pound packages

There are 120 packages. Hence

$x + y = 120$ (1)

Since $x\ \frac{1}{2}$ – pound packages weigh $\frac{1}{2}x$ pounds and $y\ \frac{1}{3}$ – pound packages weigh $\frac{1}{3}y$ pounds, we have

$\frac{1}{2}x + \frac{1}{3}y = 48$ (2)

We solve the system (1), (2) using elimination by addition. We multiply the second equation by –3 and add.

$$\begin{aligned} x + y &= 120 \\ -\frac{3}{2}x - y &= -144 \\ -\frac{1}{2}x &= -24 \\ x &= 48 \end{aligned}$$

Substituting into equation (1), we have

$48 + y = 120$

$y = 72$

48 $\frac{1}{2}$–pound packages and 72 $\frac{1}{3}$–pound packages. (*10–1, 10–2*)

55. Let x_1 = number of grams of mix *A*

x_2 = number of grams of mix *B*

x_3 = number of grams of mix *C*

We have

$0.30x_1 + 0.20x_2 + 0.10x_3 = 27$ (protein)

$0.03x_1 + 0.05x_2 + 0.04x_3 = 5.4$ (fat)

$0.10x_1 + 0.20x_2 + 0.10x_3 = 19$ (moisture)

Clearing of decimals for convenience, we have

$3x_1 + 2x_2 + x_3 = 270$

$3x_1 + 5x_2 + 4x_3 = 540$

$x_1 + 2x_2 + x_3 = 190$

Form the augmented matrix and solve by Gauss–Jordan elimination.

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 270 \\ 3 & 5 & 4 & 540 \\ 1 & 2 & 1 & 190 \end{array}\right] R_3 \leftrightarrow R_1 \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 190 \\ 3 & 5 & 4 & 540 \\ 3 & 2 & 1 & 270 \end{array}\right] \begin{array}{l} (-3)R_1 + R_2 \to R_2 \\ (-3)R_1 + R_3 \to R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 190 \\ 0 & -1 & 1 & -30 \\ 0 & -4 & -2 & -300 \end{array}\right] (-1)R_2 \to R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 190 \\ 0 & 1 & -1 & 30 \\ 0 & -4 & -2 & -300 \end{array}\right] \begin{array}{l} (-2)R_2 + R_1 \to R_1 \\ 4R_2 + R_3 \to R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 130 \\ 0 & 1 & -1 & 30 \\ 0 & 0 & -6 & -180 \end{array}\right] -\tfrac{1}{6}R_3 \to R_3 \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 130 \\ 0 & 1 & -1 & 30 \\ 0 & 0 & 1 & 30 \end{array}\right] \begin{array}{l} (-3)R_3 + R_1 \to R_1 \\ R_3 + R_2 \to R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 40 \\ 0 & 1 & 0 & 60 \\ 0 & 0 & 1 & 30 \end{array}\right]$$

Therefore

$x_1 = 40$ grams Mix *A*

$x_2 = 60$ grams Mix *B*

$x_3 = 30$ grams Mix *C* (*10–4*)

56. Let x_1 = number of tons at Big Bend

x_2 = number of tons at Saw Pit

Then

$0.05x_1 + 0.03x_2$ = number of tons of nickel at both mines = k_1

$0.07x_1 + 0.04x_2$ = number of tons of copper at both mines = k_2

We solve

$0.05x_1 + 0.03x_2 = k_1$

$0.07x_1 + 0.04x_4 = k_2$,

for arbitrary k_1 and k_2, by writing the system as a matrix equation.

$$\overset{A}{\begin{bmatrix} 0.05 & 0.03 \\ 0.07 & 0.04 \end{bmatrix}} \overset{X}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \overset{B}{\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}}$$

If A^{-1} exists, then $X = A^{-1}B$. To find A^{-1}, we perform row operations on

$$\left[\begin{array}{cc|cc} 0.05 & 0.03 & 1 & 0 \\ 0.07 & 0.04 & 0 & 1 \end{array}\right] 20R_1 \to R_1 \sim \left[\begin{array}{cc|cc} 1 & 0.6 & 20 & 0 \\ 0.07 & 0.04 & 0 & 1 \end{array}\right] (-0.07)R_1 + R_2 \to R_2$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0.6 & 20 & 0 \\ 0 & -0.002 & -1.4 & 1 \end{array}\right] -500R_2 \to R_2 \sim \left[\begin{array}{cc|cc} 1 & 0.6 & 20 & 0 \\ 0 & 1 & 700 & -500 \end{array}\right] (-0.6)R_2 + R_1 \to R_1$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & -400 & 300 \\ 0 & 1 & 700 & -500 \end{array}\right] \quad \text{Hence } A^{-1} = \begin{bmatrix} -400 & 300 \\ 700 & -500 \end{bmatrix}$$

Check: $A^{-1}A = \begin{bmatrix} -400 & 300 \\ 700 & -500 \end{bmatrix} \begin{bmatrix} 0.05 & 0.03 \\ 0.07 & 0.04 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

We can now solve the system as:

$$\overset{X}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \overset{A^{-1}}{\begin{bmatrix} -400 & 300 \\ 700 & -500 \end{bmatrix}} \overset{B}{\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}}$$

(A) If $k_1 = 3.6$, $k_2 = 5$,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -400 & 300 \\ 700 & -500 \end{bmatrix} \begin{bmatrix} 3.6 \\ 5 \end{bmatrix} = \begin{bmatrix} 60 \\ 20 \end{bmatrix}$$

60 tons of ore must be produced at Big Bend, 20 tons of ore at Saw Pit.

(B) If $k_1 = 3$, $k_2 = 4.1$,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -400 & 300 \\ 700 & -500 \end{bmatrix} \begin{bmatrix} 3 \\ 4.1 \end{bmatrix} = \begin{bmatrix} 30 \\ 50 \end{bmatrix}$$

30 tons of ore must be produced at Big Bend, 50 tons of ore at Saw Pit.

(C) If $k_1 = 3.2$, $k_2 = 4.4$,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -400 & 300 \\ 700 & -500 \end{bmatrix} \begin{bmatrix} 3.2 \\ 4.4 \end{bmatrix} = \begin{bmatrix} 40 \\ 40 \end{bmatrix}$$

40 tons of ore must be produced at Big Bend, 40 tons of ore at Saw Pit. *(10–4)*

57. (A) The labor cost of producing one printer stand at the South Carolina plant is the product of the stand row of *L* with the South Carolina column of *H*.

$$[0.9 \quad 1.8 \quad 0.6] \begin{bmatrix} 10.00 \\ 8.50 \\ 4.50 \end{bmatrix} = 27 \text{ dollars}$$

(B) The matrix *HL* has no obvious meaning, but the matrix *LH* represents the total labor costs for each item at each plant.

(C) $$LH = \begin{bmatrix} 1.7 & 2.4 & 0.8 \\ 0.9 & 1.8 & 0.6 \end{bmatrix} \begin{bmatrix} 11.50 & 10.00 \\ 9.50 & 8.50 \\ 5.00 & 4.50 \end{bmatrix}$$

$$= \begin{bmatrix} (1.7)(11.50) + (2.4)(9.50) + (0.8)(5.00) & (1.7)(10.00) + (2.4)(8.50) + (0.8)(4.50) \\ (0.9)(11.50) + (1.8)(9.50) + (0.6)(5.00) & (0.9)(10.00) + (1.8)(8.50) + (0.6)(4.50) \end{bmatrix}$$

$$= \begin{array}{c} \begin{array}{cc} \text{N.C.} & \text{S.C.} \end{array} \\ \begin{bmatrix} \$46.35 & \$41.00 \\ \$30.45 & \$27.00 \end{bmatrix} \end{array} \begin{array}{l} \text{Desk} \\ \text{Stands} \end{array}$$

(10–3)

58. (A) The average monthly production for the months of January and February is represented by the matrix $\frac{1}{2}(J+F)$

$$\frac{1}{2}(J+F) = \frac{1}{2}\left(\begin{bmatrix} 1{,}500 & 1{,}650 \\ 850 & 700 \end{bmatrix} + \begin{bmatrix} 1{,}700 & 1{,}810 \\ 930 & 740 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 3{,}200 & 3{,}460 \\ 1{,}780 & 1{,}440 \end{bmatrix} = \begin{array}{c} \text{N.C.} \quad \text{S.C.} \\ \begin{bmatrix} 1{,}600 & 1{,}730 \\ 890 & 720 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Desks} \\ \text{Stands} \end{array}$$

(B) The increase in production from January to February is represented by the matrix $F - J$.

$$F - J = \begin{bmatrix} 1{,}700 & 1{,}810 \\ 930 & 740 \end{bmatrix} - \begin{bmatrix} 1{,}500 & 1{,}650 \\ 850 & 700 \end{bmatrix} = \begin{array}{c} \text{N.C.} \quad \text{S.C.} \\ \begin{bmatrix} 200 & 160 \\ 80 & 40 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Desks} \\ \text{Stands} \end{array}$$

(C) $$J\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1{,}500 & 1{,}650 \\ 850 & 700 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3{,}150 \\ 1{,}550 \end{bmatrix} \begin{array}{l} \text{Desks} \\ \text{Stands} \end{array}$$

This matrix represents the total production of each item in January. *(10–3)*

59. The inverse of matrix B is calculated to be

$$B^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

Putting the coded message into matrix form and multiplying by B^{-1} yields

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 21 & 30 & 29 & 46 & 19 & 52 \\ 21 & 28 & 34 & 35 & 21 & 52 \\ 27 & 31 & 50 & 62 & 39 & 79 \end{bmatrix} = \begin{bmatrix} 15 & 27 & 13 & 19 & 1 & 25 \\ 6 & 3 & 16 & 27 & 18 & 27 \\ 6 & 1 & 21 & 16 & 20 & 27 \end{bmatrix}$$

This decodes to

15	6	6	27	3	1	13	16	21	19	27	16	1	18	20	25	27	27
O	F	F		C	A	M	P	U	S		P	A	R	T	Y		

(10–4)

60. (A) Let x_1 = number of nickels, x_2 = number of dimes

Then $x_1 + x_2 = 30$ (total number of coins)

$5x_1 + 10x_2 = 190$ (total value of coins)

We form the augmented matrix and solve by Gauss–Jordan elimination.

$$\left[\begin{array}{cc|c} 1 & 1 & 30 \\ 5 & 10 & 190 \end{array}\right] (-5)R_1 + R_2 \to R_2 \sim \left[\begin{array}{cc|c} 1 & 1 & 30 \\ 0 & 5 & 40 \end{array}\right] \tfrac{1}{5}R_2 \to R_2 \sim \left[\begin{array}{cc|c} 1 & 1 & 30 \\ 0 & 1 & 8 \end{array}\right] (-1)R_2 + R_1 \to R_1$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 22 \\ 0 & 1 & 8 \end{array}\right]$$

The augmented matrix is in reduced form. It corresponds to the system

$x_1 = 22$ nickels

$x_2 = 8$ dimes

(B) Let x_1 = number of nickels, x_2 = number of dimes, x_3 = number of quarters

Then $x_1 + x_2 + x_3 = 30$ (total number of coins)

$5x_1 + 10x_2 + 25x_3 = 190$ (total value of coins)

We form the augmented matrix and solve by Gauss–Jordan elimination

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 30 \\ 5 & 10 & 25 & 190 \end{array}\right] (-5)R_1 + R_2 \to R_2 \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 30 \\ 0 & 5 & 20 & 40 \end{array}\right] \tfrac{1}{5}R_2 \to R_2 \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 30 \\ 0 & 1 & 4 & 8 \end{array}\right] (-1)R_2 + R_1 \to R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & 22 \\ 0 & 1 & 4 & 8 \end{array}\right]$$

The augmented matrix is in reduced form. It corresponds to the system:

$x_1 - 3x_3 = 22$

$x_2 + 4x_3 = 8$

Let $x_3 = t$. Then $x_2 = -4x_3 + 8 = -4t + 8$

$x_1 = 3x_3 + 22 = 3t + 22$

A solution is achieved, not for every real value of t, but for integer values of t that give rise to non–negative x_1, x_2, x_3.

$x_1 \geq 0$ means $3t + 22 \geq 0$ or $t \geq -7\frac{1}{3}$

$x_2 \geq 0$ means $-4t + 8 \geq 0$ or $t \leq 2$

$x_3 \geq 0$ means $t \geq 0$

The only integer values of t that satisfy these conditions are 0, 1, 2. Thus we have the solutions

$x_1 = 3t + 22$ nickels

$x_2 = 8 - 4t$ dimes

$x_3 = t$ quarters where $t = 0, 1,$ or 2 (*10–1*)