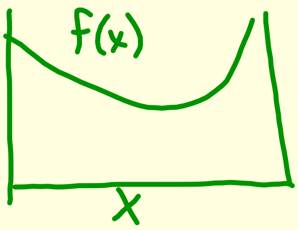
CALCULUS Unit #2 : Applications of Derivatives

7.7 Optimization Problems DAY 2

Practice 1: A non-uniform metal chain hangs between two walls that are two meters apart. The height above the ground of this chain is given by the equation $f(x) = e^{-2x} + e^{x}$, where x is the distance along the ground from the left wall.

a) Show that the minimum height occurs when $x = \frac{\ln 2}{3}$.

b) Use your GDC to find the minimum height.
Sketch your graph below.



a) $f'(x) = -2e^{-2x} + e^{x}$

$0 = -2e^{-2x} + e^{x}$

$0 = e^{x}(-2e^{-3x} + 1)$

$0 = -2e^{-3x} + 1$

$\frac{1}{2} = e^{-3x}$

$\log_e \frac{1}{2} = -3x$

$\frac{\ln \frac{1}{2}}{-3} = \frac{\cancel{-3}x}{\cancel{-3}}$

$\ln \frac{1}{2} = \ln e^{-3x}$

$\ln \frac{1}{2} = -3x$

$\frac{\overset{0}{\cancel{\ln 1}} - \ln 2}{-3} = \frac{-\ln 2}{-3} = \frac{\ln 2}{3}$

Practice 2: A small manufacturing company makes and sells x machines each month. The monthly cost C, in dollars, of making x machines is given by

$$C(x) = 2600 + 0.4x^2.$$

The monthly income I, in dollars, obtained by selling x machines is given by

$$I(x) = 150x - 0.6x^2.$$

a) Show that the company's monthly profit can be calculated using the quadratic function $P(x) = -x^2 + 150x - 2600$. [2]

$I - C = 150x - 0.6x^2 - (2600 + 0.4x^2)$

b) How many machines should be made and sold each month for a maximum profit? [2]

$P'(x) = -2x + 150 = 0$

$-2x = -150$

$x = 75$ machines

c) If the company does maximize profit, what is the selling price of each machine? [4]

$I(75) = \$\frac{7875}{75} = \$105$

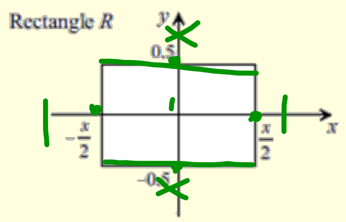
d) Given that $P(x) = (x - 20)(130 - x)$, find the smallest number of machines the company must make and sell each month in order to make **positive** profit. [4]

$x - 20 > 0$ $\quad$ $130 - x > 0$

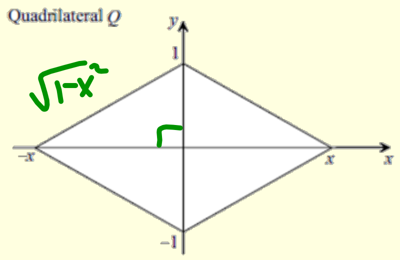
$x > 20$ $\quad$ $x < 130$

21 machines.

Practice 3:



R has vertices at $\left(\frac{x}{2}, 0.5\right), \left(-\frac{x}{2}, 0.5\right), \left(-\frac{x}{2}, -0.5\right), \left(\frac{x}{2}, -0.5\right)$.



Q has vertices at $(x, 0), (0, 1), (-x, 0), (0, -1)$, where $x > 0$.

$$P_R = 2 + 2x$$

$$P_Q = 4\sqrt{1+x^2}$$

Practice 3: $g(x) = \frac{2+2x}{4\sqrt{1+x^2}} = \frac{2(1+x)}{4\sqrt{1+x^2}}$

(b) Let $g(x) = \frac{\text{perimeter of } R}{\text{perimeter of } Q}$, for $x > 0$, *i.e.* $g(x) = \frac{0.5(x+1)}{\sqrt{1+x^2}}$.

Show that $g'(x) = \frac{0.5(1-x)}{(1+x^2)^{\frac{3}{2}}}$. Quotient Rule *[3 marks]*

(c) Find the maximum value of the ratio $\frac{\text{perimeter of } R}{\text{perimeter of } Q}$. *[5 marks]*

$g'(x) = 0$

$0 = 0.5(1-x)$

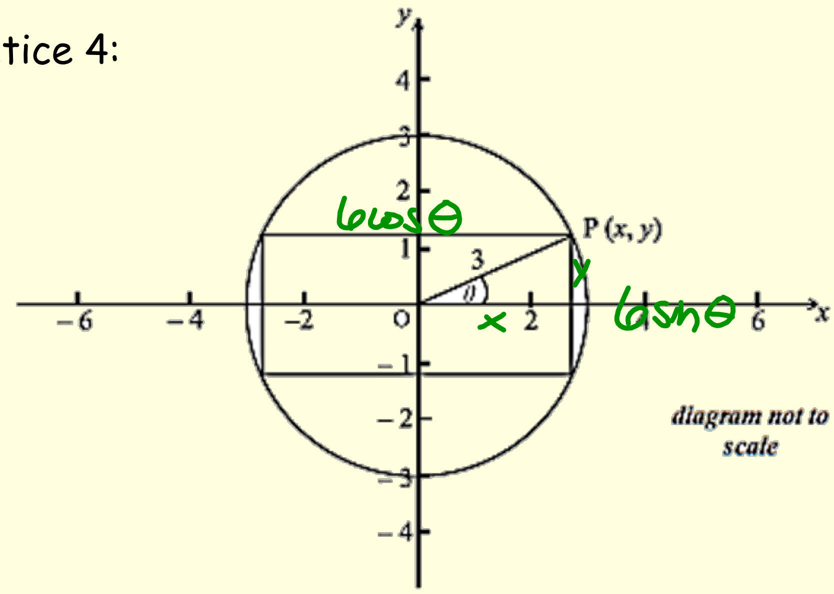
$0 = 1-x$

$x = 1$

$g(1) = \frac{0.5(1+1)}{\sqrt{1+1^2}}$

$= \frac{0.5(2)}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

Practice 4:



The point P(x, y) is a vertex of the rectangle and also lies on the circle. The angle between (OP) and the x-axis is θ radians, where $0 \le \theta \le \frac{\pi}{2}$.

(a) Write down an expression in terms of θ for

(i) x; $\cos\theta = \frac{x}{3}$ $x = 3\cos\theta$

(ii) y. $\sin\theta = \frac{y}{3}$ $y = 3\sin\theta$

$(2\sin\theta\cos\theta = \sin 2\theta)$

Practice 4:

Let the area of the rectangle be A.

(b) Show that $A = 18\sin 2\theta$.

(c) (i) Find $\frac{dA}{d\theta}$.

(ii) Hence, find the exact value of θ which maximizes the area of the rectangle.

(iii) Use the second derivative to justify that this value of θ does give a maximum.

$$A = 6\sin\theta \, 6\cos\theta$$
$$= 36\sin\theta\cos\theta$$
$$= 18(2\sin\theta\cos\theta)$$
$$= 18\sin 2\theta$$

c i) $A' = 18\cos 2\theta \cdot 2$

ii) $0 = 36\cos 2\theta$

$0 = \cos 2\theta$

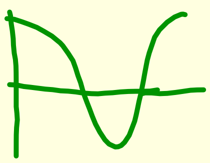
$x = 2\theta$

$\cos x = 0$

$x = \frac{\pi}{2}$

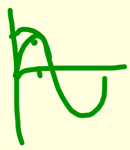
$2\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{4}$



iii) $A'' = -36\sin 2\theta \cdot 2$

$= -72\sin 2\theta$



$A''(\pi/4) = -72\sin 2(\frac{\pi}{4})$

$= -72\sin\frac{\pi}{2}$

$= -72(1)$

$= -72$

max. cc

Homework:

Finish Optimization WS and

Exercises 7Y