$uv' + vu'$

IB Math Standard

7.4 Chain Rule Further Techniques

$3(x+7)^3(x\cdot 7)$

For some functions, the chain rule must be combined with the product or quotient rule. Sometimes, the chain rule must be repeated…

1. Chain with Product

$$f(x) = x^2\sqrt{1-x^2}$$

$$f'(x) = 2x\sqrt{1-x^2} + x^2\left(\tfrac{1}{2}(1-x^2)^{-1/2}(-2x)\right)$$

$f(u) = \sqrt{u} \quad f'(u) = \frac{1}{2}u^{-1/2}$

$u = 1-x^2 \quad u' = -2x$

$$f'(x) = 2x\sqrt{1-x^2} - \frac{x^3}{\sqrt{1-x^2}}$$

2. Chain with Quotient

$$f(x) = \ln\left(\frac{x}{x^2+1}\right)$$

$$f'(x) = \frac{1}{\left(\frac{x}{x^2+1}\right)} \cdot \left(\frac{-1}{x^2+1}\right)$$

$f(u) = \ln u \quad f'(u) = \frac{1}{u}$

$u = \frac{x}{x^2+1}$

$u' = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$

$f'(x) = \frac{x^2+1}{x} \cdot \frac{-1}{x^2+1} = \boxed{\frac{-1}{x}} = \frac{-1}{(x^2+1)}$

3. Chain with Quotient

$$f(x) = \left(\frac{3x-1}{x^2+3}\right)^2$$

$f(u) = u^2 \quad f'(u) = 2u$

$u = \frac{3x-1}{x^2+3}$

$u' = \frac{(x^2+3)(3) - (3x-1)(2x)}{(x^2+3)^2}$

$= \frac{3x^2+9-6x^2+2x}{(x^2+3)^2}$

$= \frac{-3x^2+2x+9}{(x^2+3)^2}$

$$f'(x) = 2\left(\frac{3x-1}{x^2+3}\right)\left(\frac{-3x^2+2x+9}{(x^2+3)^2}\right) = \frac{2(3x-1)(-3x^2+2x+9)}{(x^2+3)^3}$$

4. Chain Repeated

$$f(x) = e^{2(3x-1)^4}$$

$f(u) = e^u \quad f'(u) = e^u$

$u = 2(3x-1)^4$

$u' = 8(3x-1)^3(3)$

$f(u_1) = 2u_1^4 \quad f'(u_1) = 8u_1^3$

$u_1 = 3x-1 \quad u_1' = 3$

$f'(x) = e^{2(3x-1)^4}\left(8(3x-1)^3(3)\right)$

$= 24e^{2(3x-1)^4}(3x-1)^3$