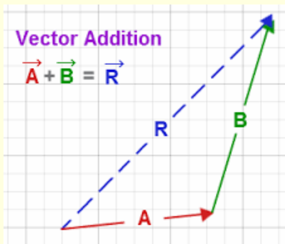
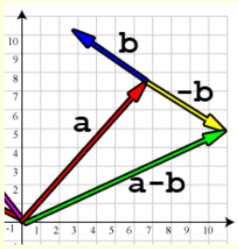
12.3 Scalar Product

Vector Operations

We have looked at several vector operations:

- addition
- subtraction
- multiplication by a scalar





Each of these operations requires

positioning vectors <u>head to tail</u>.

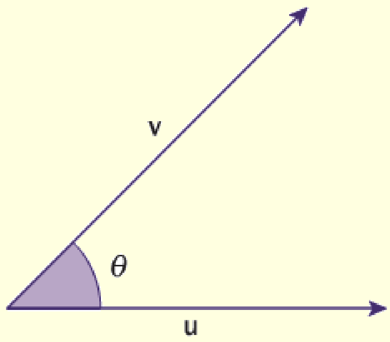
Vector Multiplication

Mathematicians have designed two different ways to multiply vectors. Each has a different method, and each result tells us something different.

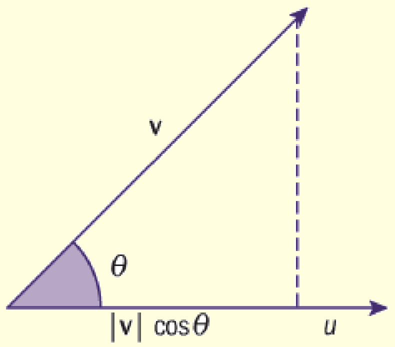
Vector Multiplication

Dot Product or <u>Scalar Product</u>	Cross Product or <u>Vector Product</u>
a•b	**a × b**
The result is a scalar quantity	The result is a vector

→ Given two non-zero vectors **u** and **v**, $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta$, where θ is the angle between **u** and **v**.



▲ To measure the angle between two vectors, the vectors must have the same initial point.



▲ $|\mathbf{v}|\cos\theta$ is called the projection of **v** in the direction of **u**.

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$$ (constant)

The scalar product of two vectors is always a scalar.

If $\mathbf{u} \cdot \mathbf{v} = 0$, then $\theta = 90^\circ$, which means $u \perp v$.

If $\mathbf{u} \cdot \mathbf{v} > 0$, then $0 < \theta < 90^\circ$, which means θ is acute.

If $\mathbf{u} \cdot \mathbf{v} < 0$, then $90 < \theta < 180^\circ$, which means θ is obtuse.

If $\mathbf{u} \cdot \mathbf{v} = \pm|\mathbf{u}||\mathbf{v}|$, then $\theta = 0^\circ$, which means $u \parallel v$.

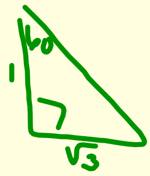
$u \cdot v = |u||v| \cos\theta$

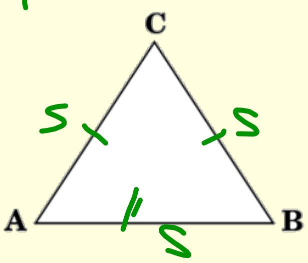
$\cos 60 = \frac{1}{2}$

1. Consider an equilateral triangle ABC.
   Find $\overrightarrow{AB} \cdot \overrightarrow{BC} + \overrightarrow{BC} \cdot \overrightarrow{AC}$.

$s \cdot s \left(\frac{1}{2}\right) + ss\left(\frac{1}{2}\right)$

$\frac{s^2}{2} + \frac{s^2}{2} = s^2$





2. Show that $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$ for any vector $\mathbf{u}$.

$|u||u| \cos 0$

$= |u||u|(1)$

$= |u|^2$

→ Given two vectors in the plane, $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$,

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2$$

In 3-D space, given two vectors, $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$,

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$$

Scalar Product
Dot Product

$$\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 \text{, where } \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},\ \boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

3. Find the dot product for each pair of vectors. Classify each angle.

a) $2i + 3j$

$i - 6j$

$= (2)(1) + (3)(-6)$

$= -16$

b) $4i - 2j + k$

$2i + j - 3k$

$= 4(2) + -2(1) + 1(-3)$

$= 3$

c) $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix}$

$= 2(-3) + -1(1) + 4 \cdot 5$

$= 13$

$\boldsymbol{v} \cdot \boldsymbol{w} = |\boldsymbol{v}||\boldsymbol{w}|\cos\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$

$$\boldsymbol{v} \cdot \boldsymbol{w} = v_1w_1 + v_2w_2 + v_3w_3, \text{ where } \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \ \boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

4. Find the angle between each pair of vectors in #3.

a) v: $2i + 3j$
w: $i - 6j$

b) $4i - 2j + k$
$2i + j - 3k$

c) $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}\begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix}$

$|v| = \sqrt{2^2+3^2} = \sqrt{13}$

$|w| = \sqrt{1^2+(-6)^2} = \sqrt{37}$

$v \cdot w = -16$

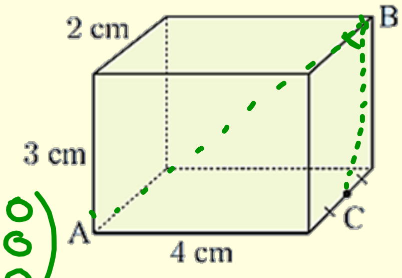
$\cos\theta = \dfrac{-16}{\sqrt{13}\cdot\sqrt{37}}$

$\theta = 136°$

$\theta = 79.9°$

$\theta = 61.3°$

5. Use vector methods to determine the measure of angle $A\hat{B}C$.



$u: \vec{BA} = \begin{pmatrix} 0-4 \\ 0-2 \\ 0-3 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ -3 \end{pmatrix}$

$v: \vec{BC} = \begin{pmatrix} 4-4 \\ 1-2 \\ 0-3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix}$

$A = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$B = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$ $C = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$

$|u| = \sqrt{(-4)^2 + (-2)^2 + (-3)^2} = \sqrt{29}$

$|v| = \sqrt{0^2 + (-1)^2 + (-3)^2} = \sqrt{10}$

$u \cdot v = -4(0) + -2(-1) + -3(-3)$

$= 0 + 2 + 9 = 11$

$\cos\theta = \dfrac{11}{\sqrt{29} \cdot \sqrt{10}}$

$\theta = 49.8°$

6. $\mathbf{a} = 4\mathbf{i} + 5\mathbf{j}$, $\mathbf{b} = x\mathbf{i} - 8\mathbf{j}$ and $\mathbf{c} = \mathbf{i} + y\mathbf{j}$.

a) Find the value of the constant x given that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

$$4x + 5(-8) = 0$$
$$4x - 40 = 0$$
$$x = 10$$

b) Find the value of the constant y given that $\mathbf{a}$ and $\mathbf{c}$ are parallel.

$$a \cdot c = |a||c|$$
$$4(1) + 5y = \left(\sqrt{4^2+5^2}\right)\left(\sqrt{1^2+y^2}\right)$$
$$(4+5y)^2 = \left(\sqrt{41}\left(\sqrt{1+y^2}\right)\right)^2$$
$$(4+5y)^2 = 41(1+y^2)$$
$$16 + 40y + 25y^2 = 41 + 41y^2$$
$$0 = 16y^2 - 40y + 25$$
$$0 = (4y-5)(4y-5)$$
$$4y - 5 = 0$$
$$y = \frac{5}{4}$$

7. Find the values of x for which the vectors $\begin{pmatrix} 1 \\ 2\cos x \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2\sin x \\ 1 \end{pmatrix}$ are perpendicular. $0 \le x \le \frac{\pi}{2}$.

$u \cdot v = 0$

$1(-1) + 2\cos x \, 2\sin x + 0(1) = 0$

$-1 + 4\cos x \sin x = 0$

$\frac{4\cos x \sin x}{2} = \frac{1}{2}$

$2\sin x \cos x = \frac{1}{2}$

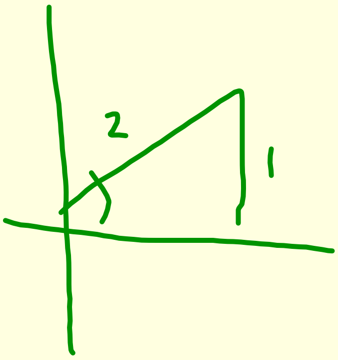
$\sin 2x = \frac{1}{2} \rightarrow (2x = b)$

$\sin b = \frac{1}{2}$

$b = \pi/6$

$\left(\frac{1}{2}\right) 2x = \frac{\pi}{6}\left(\frac{1}{2}\right)$

$x = \frac{\pi}{12}$



8. Find a unit vector which is perpendicular to both of the vectors $4\mathbf{i}+2\mathbf{j}-3\mathbf{k}$ and $2\mathbf{i}-3\mathbf{j}+\mathbf{k}$.